DEVELOPING ARITHMETIC CONCEPTS AND SKILLS

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PREFACE

This textbook is an attempt to blend arithmetic content and method for the teacher or future teacher. Because the elementary teacher usually teaches arithmetic in only one grade, content and method are organized by grades instead of by arithmetic topics. This does not mean that the teacher should know only that part of each arithmetic topic usually reserved for a certain grade. To the contrary, teachers should know arithmetic by both topics and grades.

It would be as hopeless a task to find a "typical" fourth-grade program as it would be to find a typical fourth-grade student. Consequently, no claim is made that the program described for a particular grade represents a consensus by authorities. On the other hand, many sets of contemporary texts in arithmetic and publications by experimental groups were used in the synthesis of the programs described in this book.

Four chapters develop modern number concepts for the teacher and offer an elementary presentation of the natural numbers, the integers, the rational numbers, and the real numbers. The development of the natural numbers from basic concepts of sets and the extension to integers, rationals, and real numbers is more complete than is usually found in texts of this kind. These chapters should assist the elementary and junior high school teacher in understanding many features of current trends in arithmetic.

The writers believe that method without content is a dry and unfruitful subject and that content without method does not assure learning. Evaluation, diagnosis, problem solving, helping the exceptional learner, and other topics have been made a part of the program of each grade. No steps for teaching or learning are provided. Rather, the writers have tried to present content and method as a meaningful whole for the teacher or future teacher.

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The Development of Arithmetic Teaching

It is difficult for us to imagine a school program that does not include arithmetic. Yet there have been such programs. Since arithmetic became a part of the elementary school curriculum, the nature of the material taught, as well as the methods by which it was taught, have changed drastically. Indeed, there is every reason to believe that further changes will occur as research reveals better ways of teaching and learning arithmetic. An evolving society will demand continuing changes in content.

This chapter deals with the development of arithmetic teaching from the following points of view:

A. The Arithmetic Curriculum

B. Methods of Teaching Arithmetic

C. The Learning of Arithmetic

D. The Education of Teachers

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A. The Arithmetic Curriculum

The word "curriculum" is widely used and variously interpreted. After qualifying verbiage is removed, however, most definitions come back to the very basic meaning, "that which is taught." Obviously, curriculum is related to method in that both factors function together to produce learning. Nevertheless, an effort will be made to adhere closely to the basic concept of curriculum as that subject matter which is included in a particular course.

THE ARITHMETIC CURRICULUM IN EARLY SCHOOLS

Early civilizations used systems of counting, and some of them developed fairly elaborate systems of computation. The abacus, which is still a basic instrument in certain types of quantitative work, is believed to date back as far as the sixth century B.C. This device was adequate for the needs of its users; hence, for many centuries, it underwent very little change.

Obviously, if systems of numeration and computation were used, they had to be taught. This was usually accomplished through practice, with a certain amount of tutoring. And even this rudimentary arithmetic was taught only to those who needed it in their work.

Generally, arithmetic was relegated to a minor position in the schools of the Ancient World. For example, Smith[1] says of ancient Greece, "Due to the cumbersomeness of their numeral system and their antipathy to commercial and other practical occupational pursuits, training in arithmetic appears to have been very meager, extending probably not appreciably beyond simple counting with the aid of the fingers and an abacus."

The European merchants who brought the Hindu-Arabic number system to Europe saw computation as a very useful art. It can be assumed that they taught the practical aspects of this science to their associates and apprentices. The schools of Britain during the seventeenth century concentrated on those phases of arithmetic that would be useful to tradesmen. Efforts to interest "young gentlemen" in studying arithmetic were largely unavailing, so that, until the eighteenth century, the study of arithmetic was based almost exclusively on the criterion of usefulness.

[1] William A. Smith, *Ancient Education* (New York: Philosophical Library, 1955), p. 134.

THE ARITHMETIC CURRICULUM IN AMERICAN SCHOOLS

The educational concepts that were dominant in England at the time America was being settled also became dominant in America—simply because these ideas were the only ones the settlers knew. Thus a purely classical type of education was established in America, with the Latin grammar school occupying a position of great prominence at the secondary level. It is hard to imagine a community as beset with dangers, discomforts, and hardships as was a typical New England village during the Colonial period. It is even harder to imagine these people establishing schools where the chief aim was to teach Latin, Greek, and sometimes Hebrew—this in a new world where there was an urgent need for surveyors, navigators, bookkeepers, and many other workers in specialized areas.

Catalogs indicate that some of the larger Latin grammar schools did give passing attention to the rudiments of arithmetic, but it received far less time and attention than did the classics.

With the founding of the first academy in 1751—it was sponsored by Benjamin Franklin—the place of arithmetic in the curriculum changed drastically. Franklin was anxious to see students get a type of training which would have practical application in their society, and certainly there was great need for arithmetic.

Some of the textbooks used in the academies tell a great deal about the arithmetic curriculum of that era. One of these was the *Schoolmaster's Assistant*,[2] originally a British publication, but "adapted to the United States." Included in the text are several recommendations, some of which were written by teachers at academies. The author of this text believed in the direct approach. The second page of text materials contained an addition and subtraction table, followed by a multiplication table. Thus a student, in order to advance beyond page 2, must know facts now taught as late as fourth grade. This book is essentially a compendium of rules and tables, with practically no attention given to understandings.

The public high school first appeared early in the nineteenth century but did not become a major element in the educational system until after the Civil War. With it came specialized mathematics courses, such as those now found in the high schools, but arithmetic still occupied a prominent position in both elementary and secondary curriculums.

[2] Nathan Daboll, *Schoolmaster's Assistant* (E. and E. Hasford, 1821).

Franklin's interest in practical training, however, somehow became obscured by the "training of the mind" or mental discipline idea. Thus, until well into the twentieth century, many arithmetic problems were taught solely because they were difficult and hence exercised the mind. For example, one arithmetic book[3] had the students change .821437437 . . . to a common fraction. The answer is $\frac{102577}{124875}$.

Another manifestation of the mental discipline concept was the emphasis given to mental arithmetic. One textbook in mental arithmetic[4] included this problem. "A lady, being asked the hour of the day, replied that $\frac{2}{3}$ of the time past noon equaled $\frac{4}{5}$ of the time to midnight, minus $\frac{4}{5}$ of an hour; what was the time?" When one remembers that such exercises were to be done "in the head," one can appreciate the unhappy lot of the student during the period when the concept of mental discipline dominated the teaching of arithmetic.

THE MODERN CURRICULUM IN ARITHMETIC

As the preceding material shows, the modern curriculum in arithmetic has a long and honorable history. No ultimate answers have been reached, however, and the same developmental processes continue to function.

Generally, the elementary arithmetic program in the modern school is built around several major concepts.

The number system. For many years, it was considered adequate if young children learned to rattle off the numbers from 1 to 100, even though it might be a meaningless chant. Now, children are taught a great deal about our number system—the fact that we use a base of 10, the fact that 16 means six-ten, and so on. The structure of numbers gets a great deal of attention. And with the help of teaching materials, the teacher hopes to build a concept of number so that even the most elementary arithmetic experiences will contribute to understanding.

The fundamental operations. The processes of addition, subtraction, multiplication, and division have long served as the focal point of much arithmetic teaching. But the approach used in such teaching has been drastically revised. A few decades ago, for example, when a student

[3] Joseph Ray, *New Higher Arithmetic* (New York: American Book Company), p. 119.

[4] Edward Brooks, *New Normal Mental Arithmetic* (Philadelphia: Christopher Sower Company, 1873), p. 160.

came to the "multiplication tables," he didn't move until he could recite the facts included in it. Thus, he might spend weeks on end drilling on an exercise which was, to him, rote memory. Now it is recognized that these facts have meaning and hence can be taught meaningfully. Also, instead of learning all the facts in one prolonged session, he studies part of them one year and part another so as to lighten the drudgery.

This does not imply that the modern student studies fewer facts than did his predecessor. The big change is that he now studies them as concrete processes that have meaning, and he takes a few at a time. In either case, the goal is mastery of the facts—addition, subtraction, multiplication, and division. But the content for each year has been changed so as to make learning the facts a gradual process.

Measurement. The modern arithmetic curriculum includes various topics associated with measurement. Now, however, the emphasis is on measures that have meaning to children—simple money units, length, weight, volume, and others. This is in marked contrast to earlier practices when students labored mightily with units which were meaningless to them. Consider, for example, the student of 1830, who learned that 20 grains make 1 scruple or 3 scruples make 1 dram. Many such measures have disappeared from the arithmetic curriculum because they are too specialized for general use. The measures presently included in arithmetic books have, by and large, met the test of practicality.

Social usage. One of the big changes in arithmetic content is that material is presented at a time when it should have meaning to children. Problem situations are based on children's games, school and home activities, or other such areas. Texts show children who look like children and who are engaged in typical childhood activities. Generally, the problems are designed to yield realistic results. There is a vast difference in this respect between our modern books and earlier ones presenting such items as the old standard, "If a hen-and-a-half lays an egg-and-a-half in a day-and-a-half" type of problem.

THE CHANGING AIMS IN ARITHMETIC TEACHING

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method was adequate for his immediate needs, he made no particular effort to improve upon it. Similarly, the fruitseller who sketched a crude abacus in the dust deposited on a flat surface was using arithmetic at a level such that she had no incentive to work toward improvement. Consequently, for many centuries, rudimentary arithmetic was taught only to those who had specific need for it, and generally their learning was at a relatively low level.

Then ideas about the place of arithmetic in education changed. The concept of mental discipline displaced practicality as the purpose of teaching arithmetic. In early American schools, the shift was a major one. The transition was gradual but a few names stand out. Edward Brooks, who wrote a widely accepted text on mental arithmetic,[5] gave credit to Warren Colburn for his work in introducing mental arithmetic. Of mental arithmetic, Brooks said, "When properly taught, it gives quickness of perception, newness of insight, toughness of mental fibre, and an intellectual power and grasp that can be acquired by no other elementary branch of study." Thus it was that, for many decades, teachers sought to develop young minds as though they were developing muscles—through vigorous exercise.

With the gradual rejection of mental discipline as the dominant goal in the teaching of arithmetic, social utility began to move into the picture. This did not mean a reversion to the goal of practicality. Rather, effort was directed toward teaching children as children instead of miniature adults. The activities used in teaching arithmetic were activities which children would normally enjoy. Teaching situations were real; concrete materials were extensively used. This change of approach was criticized as sugar-coating arithmetic or causing loss of arithmetic subject matter. The new emphasis, however, has brought about a major change; in the modern school, many children actually enjoy arithmetic.

In commenting on arithmetic as it was taught under the mental discipline approach, one veteran teacher remarked that, when she was in the elementary grades, arithmetic was studied early in the day. Ostensibly this was done because minds were fresh, but the real reason was that the arithmetic class was an unpleasant experience which everyone desired to complete as early as possible.

Social utility as an aim in the teaching of arithmetic is still important. Recently, however, arithmetic for understanding has been given added

[5] Brooks, *op. cit.*, p. 3

emphasis. Thus, children are being taught to understand many operations which for years were taught as rules, formulas, and procedures. For example, the process of borrowing in subtraction was long taught as an abstract procedure. It is now presented to students as a logical method of number treatment which, when understood, can be carried out quite successfully by most students without the use of rules and step-by-step methods.

It can be seen that the goals toward which teachers strive in the teaching of arithmetic change over the years. As long as society is changing, aims of teaching must change.

THE EVOLVING CONTENT OF ARITHMETIC

Just as aims have changed, so has the content of arithmetic changed. Emphases—concepts of the importance of certain processes, for example—are constantly being revised.

As an example, one might consider the role of counting in arithmetic. In earlier American schools, the teacher's first goal was having all her students learn to count by rote to 100. This was merely a matter of saying number names in a sort of chant which had little meaning for anyone. More recently, the emphasis has come to be placed on rational counting—counting things. Indeed, it is an open question whether rote counting is counting at all, since no things are being enumerated. Too, the idea that "this must be learned before we go to anything else" has largely been abandoned.

There have been certain changes regarding the use of cardinal and ordinal numbers—notably, that the two are different and are not of equal difficulty. Incidentally, a cardinal number tells how many ("I have four brothers"), whereas an ordinal number tells the position in an order or sequence ("I am the fourth child in our family"). The latter concept is generally considered to be more difficult than the former; hence it is developed fairly slowly.

One major change regarding arithmetic content is the attention being given to the structure of numbers. Textbook writers now consider a knowledge of the number system to be so important that they give some work on it at each grade level throughout the elementary school. That 42 means 4 tens and 2 ones; that, in 111, the progression from right to left involves a multiplier of 10—these facts and many others are now presented in a way which shows that there are reasons for doing what is done. These reasons can be seen and understood. Terms

like *place value* and *place holder* become part of the student's arithmetic vocabulary at an early age, since these terms describe certain vital characteristics of the Hindu-Arabic numeral system.

In teaching the four fundamental operations (addition, subtraction, multiplication, and division), the biggest single change probably is the current emphasis on understanding why things are done as they are. Also, learning a set of facts is now broken down into smaller assignments. For example, most arithmetic texts introduce the addition facts a few at a time, thus scattering the task over a period of two or three grade levels.

Common fractions, long the mainstay of the mental discipline advocates, have been given less and less attention in recent years. One reason for this reduced emphasis was the low social utility of many of the common fractions formerly included in elementary arithmetic. Research studies pointed out that certain common fractions were useful; these were the ones that were emphasized in arithmetic. More recently the concept of common fractions as rational numbers has received attention in the upper elementary grades. This is a departure from the social utility criterion but is based upon the development of an understanding of fractions as numbers. Also, it is generally true that decimal fractions are easier to use than common fractions, since the decimal fraction has a "built-in" common denominator (tenths, hundredths, and others). Increased attention to decimal fractions has resulted in added emphasis on the per cent concept.

Problem solving is still considered to be a vital part of arithmetic—and children still have trouble with it. Probably the most significant recent change in problem solving is the tendency to use meaningful problem situations. Problems about pets, planning a vacation trip, or playing dominoes are far more effective in getting and holding a student's attention than problems about "How much will 6 barrels of fish cost . . . ?" Also, in problem solving, the modern textbook writer tries to use realistic data that will yield reasonable results.

Some simple geometry has been introduced at the upper elementary level. Usually this is built around problems based upon the room or playground, with a minimum of attention given to learning definitions. A certain amount of attention has been given to units of measure, but fewer units are now included. Efforts have been made to limit the work in measurement to useful units. This criterion has eliminated many of the measures that used to figure prominently in arithmetic work.

The study of arithmetic content continues to interest several groups. The work of some of these groups has led to the introduction of many new topics, such as set theory, number bases, and elementary statistics, into certain elementary programs. The mathematical nature of the number system and number operations are being considered in the upper elementary grades.

CHANGES IN GRADE PLACEMENT OF TOPICS

The idea of teaching children grouped into grades is relatively new in education, since formerly teaching was done, however hastily, by working with individuals. Consider the plight of the teacher who tried to teach, on an individual basis, 72 students, ranging from beginners to high school level.

With the coming of the graded school, the situation in arithmetic became quite confused. Nobody knew, for example, what should constitute fourth-grade arithmetic or how it should differ from fifth-grade arithmetic.

One approach was to leave a text unclassified as to grade level. For example, David Eugene Smith published an arithmetic text in 1904 which was entitled simply *Primary Arithmetic*.[6] He explained that this text was adequate for the first four school years. This eliminated the necessity of placing topics or processes specifically in a single grade.

When grade placement of topics finally became widely accepted, it was essentially a rule-of-thumb operation, since little or no research had been done regarding difficulty of topics. Generally, the tendency was to place topics at too low a level; that is, at a level where they would be extremely difficult. For example, some authors introduced inches, feet, and yards very early in the first grade. This would be very confusing for first-grade students.

Within the past few decades, however, there has been a great deal of research regarding difficulty of topics. Such work has resulted in moving many arithmetic processes into higher grade levels than was formerly the case. For example, in the study of inches, feet, and yards, a modern arithmetic program might introduce these units separately, with the inch at first-grade level, the foot at second, and the yard even later. Generally, the prime factor in grade placement of topics has been the level at which a student could be expected to understand the process.

[6] David Eugene Smith, *Primary Arithmetic* (New York Ginn and Company, 1904).

Recently, there has been some evidence of a reversal of the trend just described. Arithmetic texts have been criticized as being too easy; it is said that some of the processes should be introduced at an earlier grade level. If research indicates that this contention is valid, presumably some topics will be moved to a lower grade level. Such changes may be seen in some arithmetic texts.

Arithmetic is a living, growing area of study. New topics, processes, and operations are constantly being developed and added to the body of subject matter. With each such addition comes the question of where to place it. Only research can determine the difficulty of the new topic. Hence, grade placement should be at such a level that mastery of the new material can reasonably be expected of students.

CHANGES IN SEQUENCE OF TOPICS IN ARITHMETIC

The sequence of topics, and the way in which the teaching of an operation evolves, frequently illustrates a basic conflict between logic and psychology. For many years, textbook authors in arithmetic took the position that a logical approach was the best approach in teaching.

This can be illustrated by the multiplication tables, long the key item in teaching multiplication. It was logical to use this approach: "Now we shall master the multiplication facts." Generally, a student stayed on this awesome table until he could recite all of the facts from 1×1 through 12×12, and he didn't take up anything else during the time he was on the tables. Threats, cajolery, corporal punishment—any of these that "motivated" a student to learn his tables would be used. And at the end of this operation, the student knew, for the moment, all of the facts in multiplication.

With the development of psychology, however, the learning process came to be better understood. Gradually, the logical has given ground to the psychological. Now, students learn the multiplication facts in "families" or groups. After the family of facts is learned and used, the student works at some other type of operation. Many arithmetic programs delay complete mastery of the multiplication facts until fourth grade, although some of the facts are learned in the second and third grades.

Another effect of modern psychological concepts of learning is the emphasis on reteaching. Although it might be logical to assume that "once learned always known," parents and teachers have observed that most teaching must be repeated over and over again. Thus, in teaching

multiplication facts, most teachers will reteach the ones that were learned earlier, then introduce new ones.

Another factor that has influenced the sequence of topics is the previously mentioned research on difficulty of topics. In earlier years, for example, students were taught from the first to write division remainders as common fractions. As information about the learning process and the difficulty of operations became available, however, students were taught to write remainders simply as remainders. Now, the common-fraction way of writing this quantity is usually delayed until the upper elementary grades.

INCREASED EMPHASIS ON SOCIAL ARITHMETIC

As was mentioned earlier, arithmetic content underwent certain changes in keeping with growing knowledge about the learning process. Arithmetic, however, has always been affected to some degree by social conditions. For example, in the area of measurement, communities would establish units in keeping with their own needs; frequently, these units would be entirely different from those in a neighboring community. Trade between these societies, however, would ultimately bring the two systems of measures into harmony with each other.

Indeed, early arithmetic was probably established on the basis of social need alone. The sheepherder who used a one-pebble-for-one-sheep system of enumeration was using a system that met his immediate social needs.

This pattern changed considerably, of course, during the period when mental discipline was the dominant criterion in determining content. Obviously, a problem could exercise the mind equally well with, or without, social significance. Little attempt was made to consider the problems in a local area, since the textbook problems were well-suited to the job of "mind-stretching." Consider this problem: a man lost $\frac{2}{5}$ of his hens and found that if he sold $\frac{2}{3}$ of the remainder at cost, he would receive 40 dimes, but if he kept 15 and sold $\frac{2}{3}$ of the remainder, he would receive 20 dimes. How many bens did he have?[7] When it is added that this problem was to be done without the help of pencil and paper, one can readily see that difficulty was its chief merit. Certainly a man could stay in the chicken business a long time without encountering need for such mental gymnastics.

[7] Brooks, *op. cit.*, p. 167.

The modern program of arithmetic instruction has evolved on the basis of logic, psychological laws, and social need. Writers of textbooks constantly strive to present problem situations that are real to the age group under consideration. Thus, children's games are the basis for many problems.

Nevertheless, it remains for the individual classroom teacher to show that arithmetic functions in her own community. Only she can base realistic problems on wheat in a wheat-producing area—or cotton in a cotton-producing area. One teacher from a coastal community has pointed out that student interest in his fifth-grade arithmetic class soared when the group worked problems about shrimp or fish. Problems about growing apples or mining coal—both foreign to the community—aroused little response. Whatever the efforts of textbook producers, therefore, only the teacher can make full use of social factors in the teaching of arithmetic.

Our ideas regarding "social utility" are undergoing a gradual expansion. For example, an understanding of abstract number concepts is coming to be recognized as having social value. The content of many arithmetic programs reflects this re-interpretation of the term.

THE CHANGING ROLE OF MATERIALS IN ARITHMETIC TEACHING

Concrete objects have completed a cycle in arithmetic work and arithmetic teaching. The one-to-one correspondence concept found in early societies was based entirely upon the use of objects as counters. Also, the abacus represents an extension of this same general approach to arithmetic.

During that period of educational history when mental discipline was the dominant philosophy, material aids were suspect, since the teacher operated on the premise that arithmetic was by its very nature abstract. Furthermore, any attempt to present arithmetic in any other light made it easier, and that was undesirable. Consequently, there was little use of concrete objects.

In recent years, however, educators have realized that, from the very first, instruction in arithmetic is somewhat different from instruction in other subjects. In reading, for example, "the" has only one symbol and one meaning. But beginners are taught "three" and "3" about the same time, in effect using two languages. Then comes the difficult task of developing an understanding of the quantity described by this word and symbol.

It has been found that young children make their best progress when numbers are introduced by use of concrete objects—pencils, books, cardboard disks, even children themselves. The same method is valuable in teaching certain phases of the fundamental operations. For example, in learning the addition fact $3 + 2 = 5$, a student can check himself by counting three pencils in one stack, two in another, and verifying that combining these groups yields a single group of five pencils.

A good teacher realizes when a particular student has worked long enough with manipulative materials and is ready to move to semiconcrete representations. Thus, instead of having him use three pencils and two pencils in groups, she shows him pictures of such groups of pencils. This is somewhat more difficult, since he doesn't actually handle the pencils.

Only the teacher can work with students in the concrete phase; obviously, textbooks are not adapted to such teaching. As students move into the semiconcrete phase, however, the pictorial representations in texts or workbooks are very valuable. Also, as will be pointed out in detail later, there are many films, filmstrips, manipulative devices, and other visual aids that can assist the teacher at the semiconcrete level.

It is as true now as it was during the mental discipline period that arithmetic basically deals with abstractions. But it is also true that abstract concepts have little meaning for small children. Hence, the modern teacher starts young children at the concrete level but leads them as rapidly as she can to the semiconcrete level and ultimately to abstract treatment of arithmetic.

NEW EMPHASIS ON EVALUATION

For maximum efficiency in arithmetic teaching it is necessary that the pupil, the teacher—and at intervals, the parents—know what kind of progress is being made. Hence, evaluation is a continuing process. Many phases of arithmetic work are used in evaluation—homework, seatwork, study habits, and others. Testing however, is the backbone of the evaluation process.

There is not and cannot be a replacement for the teacher-made test. Only the teacher can judge when a test is needed or the type and extent of testing needed. Only the teacher can use the results of a test in the correction of defects in the learning process. Hence, the tests, written

or oral, that are devised by the teacher, analyzed by the teacher, and used by her in the teaching process can never be replaced by tests ordered from a publisher.

Considerable attention is being given to testing in arithmetic. For example, the use of inventory or readiness tests, usually oral, for beginners is recommended. Frequent use of diagnostic tests, designed to locate specific difficulties, is also recommended. And there is no substitute for frequent testing as a means of providing work incentives for students.

Standardized tests for use in arithmetic teaching are numerous, and they do have a place in the teaching process. Many teachers, for example, like to use a standardized test near the end of the school year to appraise the standing of their class in terms of the national norms. Others like to use such tests at the beginning and end of the school year as a method of measuring progress. School administrators sometimes use standardized tests to determine grade level for new students. And there are some excellent standardized diagnostic tests on the market. Certainly, arithmetic teachers desire all the help that is available to them in the process of evaluating progress—including standardized and locally prepared tests.

Many schools require teachers at all grade levels to give students marks. This can be an especially thorny problem for those who work with small children. Frequently, a school-wide pattern is used in such marking, which spares the teacher the job of setting up a general method. Nevertheless, the teacher should keep several factors of marking in mind: (1) she should be sure she knows where every mark came from so that she can make appropriate explanations where necessary; (2) she should be aware that she is marking people as well as arithmetic; (3) she should remember that, frequently, marking can be used as an effective teaching device. Of course, no general plan of marking can be recommended. Frequently, local policies and customs dictate the patterns used in a particular community.

B. Methods of Teaching Arithmetic

Why give attention to methods of teaching? Isn't teaching just "doing what comes naturally?" If, in a given situation, one person logically qualifies as a teacher and others, with equal logic, fill the role of learners, isn't teaching-learning the inevitable result?

Most people can, on the basis of their own experience, cite instances that negate the position just mentioned. Many will recall teachers who were well versed in subject matter but who were very poor teachers. Some teachers consider that telling is teaching, although learning is an active, rather than passive, process. Others can't simplify explanations sufficiently. Then there are the chalkboard scribblers, the mumblers, and many others—frequently strong on subject matter but poor teachers nonetheless.

Teaching arithmetic is essentially a building operation, with each grade level adding to the structure. But each builder need not evolve his building techniques from the grass-hut stage. Rather he is trained to profit by the research of others who preceded him. And thus it is with each arithmetic teacher. A great deal of investigative and developmental work has been done regarding methods of teaching. Isn't it only sensible for a contemporary teacher to profit from such research? If some methods have been shown to be better than others, isn't it logical that a modern teacher spare herself and her class the ordeal of working with the poorer methods?

But aren't good teaching methods essentially the same in all areas? This position, frequently taken by students, is analogous again to the building industry. Does it follow that a good plumber would be a good carpenter, since both work on the same structure? Teaching arithmetic differs from teaching social studies, language arts, physical education, or music for the obvious reason that in each area the teacher is working toward certain specific goals. And the goals differ for the different subject areas. Indeed, many students recall certain of their earlier teachers who were outstanding in some areas but mediocre in others.

The teaching and learning of arithmetic have several unique features. One, previously mentioned, is that each year of arithmetic work is based squarely on that of the previous year, so that gaps in the learning process must be closed. If this is not done, the student is crippled in subsequent work. A second characteristic of the learning of arithmetic is that exact quantitative thinking is relatively new to a school beginner. Hence, the teacher is faced with the job of teaching the student something for which he, at the time, sees little need. A third feature is that, quite early in the process of learning arithmetic, students are called upon to learn number facts (such as $2 + 3 = 5$). Learning material of this type of exactness is new to small children.

It is apparent, then, that in arithmetic, a child is called upon to do a type of learning which, to him, is new and different. It follows that special techniques of teaching are needed in carrying out this operation.

EARLY METHODS

An elderly man, describing his experiences in school, recently remarked that about all he ever heard his teacher say was "Next." Essentially, the teacher heard each pupil recite his lesson. Little actual teaching was possible, since all pupils at all grade levels were taught all subjects—individually. Later, when the technique of grouping students by grades became standard practice, more time was devoted to teaching and less to "hearing lessons."

During the period when mental discipline was the dominant goal in arithmetic, the methods of teaching were the ultimate in simplicity. In the field of mental arithmetic, discipline reached its peak of perfection. A typical teaching procedure might be as follows: (1) the teacher would read the problem to the class. The students listened intently, of course, since they had no books or writing materials to help them remember the problem. (2) A student would be called upon to work the problem. (3) This student would rise, repeat the problem, and give the solution. (4) Mistakes would be corrected by the teacher or the class. Other procedures were recommended, but all closely paralleled the one just described. No one could accuse this method of having social significance, since the conditions imposed were far from reality. Also, many of the principles of good teaching as they are now recognized were violated by such a procedure. But there can be no doubt that it exercised the mind.

During the period of mental-discipline domination, the problems and exercises used in arithmetic texts were frequently very difficult. Oddly, however, little attention was given to the basic principles of arithmetic. The question of "why we do what we do" seldom received any attention. In presenting a new topic, textbook writers usually followed this pattern: (1) a few illustrative examples would be worked; (2) a rule would be stated (frequently in italics); (3) long columns of exercises would be given. Sometimes, at the end of the exercises, there would be verbal problems applying the operations under study. Rules were usually presented with little or no mention of the logic upon which they were based.

Modern studies in psychology have shown that when rules are

learned without any basis of understanding, the learning is usually temporary. This might, in part, show why teachers a few generations ago frequently found it necessary to "turn a class back"; that is, go back and restudy material already covered. Possibly by the latter part of the school year, the rules and manipulative patterns learned earlier in the year had been forgotten.

Another feature of earlier arithmetic teaching was the little attention given to correlating learnings in the various areas. Rarely did anyone mention that arithmetic had much to offer in understanding one's environment. Ordinarily, the teaching was dominated by the text, and texts can be correlated with a classroom situation only with assistance from a resourceful teacher. Hence, many problems were artfully contrived—to be difficult but not necessarily to be meaningful. Consider, for example, problems about paving roads with chewing tobacco! Or compound interest problems involving loans for eight years, compounded semiannually, for fifth-graders.

Generally, therefore, arithmetic was something that was studied in arithmetic class, then put away for the day. If it had any carry-over value into other subject areas, such values were usually ignored.

MODERN METHODS

Specific methods of teaching arithmetic in the modern school are presented later in this book. But some general characteristics of such methods are worthy of note.

One of the biggest and most significant changes is the present-day emphasis on understandings. This is in sharp contrast with earlier methods, where a great deal of attention was given to rote memorization of rules and formulas. This does not imply, of course, that such timesavers as formulas are no longer used. Rather, the sequence of learnings has been changed. Currently, an attempt is made to evolve formulas on the basis of understandings of processes. Formulas (area = length × width, for example) are seen as summary statements of principles that are understood and accepted—not as meaningless sets of symbols to be memorized.

A case in point is in the division of one common fraction by another. For many years, students learned to recite: "Invert the terms of the divisor and proceed as in multiplication," with practice work—a great deal of it—following. Usually, no questions were asked or answered as to why this was done—it was just done. Modern texts usually give

several pages of explanation and illustrative material on this process, and from these a pattern of operation evolves, the pattern being the rule just cited. The hoped-for result is that the learning will be based upon understanding.

Another technique widely used in present-day arithmetic is the individualization, so far as is feasible, of instruction. In a sense, America's schools have completed a cycle as far as the attitude toward the individual is concerned. Early schools were ungraded, the teacher giving fleeting attention to each pupil. With the coming of the graded school, some teachers tended to move all pupils along at the same rate. This, of course, meant that the rate of progress was geared to students of average ability. The rapid learner was being held back; the slow learner was being left behind.

Now, however, there is general acceptance of the principle that learning is an individual process. It follows logically that attempts be made to teach accordingly—which completes the cycle just mentioned. Practical considerations, however, limit the teacher in her efforts to work with individuals. Many techniques are being used as next best. Some of these are small-group procedures, remedial work for slow learners, enrichment work for rapid learners, and whenever possible, direct contact with individual students on special problems. Some attention is being given to teaching machines that would help the teacher in individualizing instruction.

Modern arithmetic programs accept the fact that numbers are basically abstract. Further, they recognize that abstractions are difficult for small children. Hence, a beginner is no longer called upon to do abstract or rote counting to 100 before other topics are introduced. Rather, he works with concrete materials in the early phases of his number study. For example, "six apples" means a great deal more to a small child than does just "six." Hence, in their first number work, students use objects or counters until they gain some facility with numbers. A good arithmetic teacher moves students away from the concrete and toward the abstract as rapidly as she can. Some types of work, such as common fractions, however, are frequently taught with the help of concrete materials even at the middle and upper elementary levels.

MEANINGFUL CONTENT

Numerous changes have been made in the content of arithmetic books during the past several decades. Generally, the goal has been

including problem activities built around situations that are meaningful for children. Textbook writers have tried to shift the emphasis from the world of the adult to that of the child.

This change in content can be illustrated by certain topics that are or were used as bases for problems. Around the turn of the century, the following situations were used in various primary arithmetic texts: growing wool, shipping freight by rail or barge, operating life-saving stations for ships, and buying farm land. These and other similar topics, although well-suited to problem working for adults, had little meaning when presented to small children. On the other hand, modern lower-elementary texts usually base problems on such topics as planning a party, working with money, going to the circus, and other types of activities which would be real to most of the children of the grade level in question. This does not mean that the content has been "watered down"; rather, the applications of number work are based upon children's normal activities.

Another change of content has been a de-emphasis of abstractions. Of course, numbers are by their nature abstract and ultimately are dealt with as abstractions, but problems based upon remote or hard-to-visualize situations get far less attention than was formerly the case. Consider the previously cited "If a hen-and-a-half lays an egg-and-a-half in a day-and-a-half. . . ." Such a situation defies the visualizing ability of even the most astute dreamer. Also, earlier programs gave considerable attention to such problems as "24 is $\frac{3}{4}$ of how many times $\frac{2}{3}$ of 12?"

Such teaching material is very different from that incorporated in most modern programs. Today's teacher finds it possible to present a great deal of arithmetic through play activities. Teachers' editions of textbooks and teachers' manuals to accompany the various texts offer much help to teachers in setting up such situations.

The school store typifies the type of teaching just mentioned. This, of course, is a play store, but many classrooms set them up to look quite realistic. And under the leadership of a skilled teacher, a class can work on an amazingly diverse group of learning activities in its school store.

THE ROLE OF DRILL

For many years, the teaching of arithmetic involved a great deal of drill. Textbooks frequently included entire pages of exercises, introduced with such terse directions as "Add the following," "Subtract as indicated," or simply "Multiply." Although texts did vary, it was

generally true that a large percentage of course time was devoted to such drill exercises.

It is still accepted as a basic principle of learning that "knowledge of facts" is transient unless there is an extended period of exercise, but there have been several changes in the application of this principle. First, the drill exercises are broken into smaller units, so as to minimize the student's feeling that he is being overwhelmed with hours of repetitive work. Second, many ingenious methods are being used to vary the routine for the student. Third, extensive research as to relative difficulty of various types of exercises has enabled writers of textbooks to observe an increasing order of difficulty in exercises. Fourth, much effort has been given to developing techniques which would make drill more palatable to students. For example, games and activities are widely used in the learning of number facts. Fifth, in the modern arithmetic program, drill does not precede understanding of the processes being studied.

Hence, it would be incorrect to say that drill is no longer used. Rather, attention has been given to the development of a variety of methods by which drill can be effectively utilized.

PROBLEM SOLVING

Some students have a great deal of trouble with one phase of arithmetic, problem solving. Yet this is generally considered to be a vital part of the course.

As has been mentioned previously, some major changes are taking place in the types of problems students are asked to work—specifically, efforts are being made to base problems on the world of the child rather than that of the adult. Indeed, many of the problems in earlier texts didn't seem to come from either world.

There are various explanations of the difficulty many students have with problem solving. One factor, of course, is the need to read. Since many students are poor readers, it follows logically, that they should have trouble in problem solving.

Many teachers, however, can cite cases of students who read well but who still progress slowly on problem work. A possible explanation is that problems usually contain expository rather than narrative reading matter. A student must not only read with comprehension, he must arrive at a course of action to be used in the solution of the problem. Consequently, some students who can read a story acceptably, find it

difficult to read at the high level of comprehension required in problem solving.

During the past several decades, numerous attempts have been made to arrive at a method of step-by-step procedure which would simplify the solving of "word" problems. These have proved to be of doubtful value. After all, if problem solving is reduced to a stereotyped procedure, one may wonder whether it should even be called problem solving.

Nevertheless many teachers have found certain types of approaches helpful in teaching problem solving. Some of these are (1) drawing diagrams, (2) studying the vocabulary, (3) estimating answers, (4) using dramatizations, (5) "talking through" the problem, (6) using concrete objects. Yet despite these and various other techniques, it is still quite common to hear a student remark, "I can do the operations, but I have trouble with problem solving."

THINKING IN ARITHMETIC

We are frequently told that ours is a quantitative world. Yet man in his primitive state had little occasion to think quantitatively. He developed such techniques only when they were needed, and they were usually rudimentary in nature.

A limited amount of number development occurs in the early life of a child. In his preschool world, he encounters few situations that require him to do quantitative thinking. He may know that his favorite television show starts at 6:30; he may know certain counting games, or he may successfully count by rote to 100. Few preschool children go very far beyond this level of operation, however, mainly because adults do their quantitative thinking for them. Thus an elementary teacher is faced with the task of creating awareness of a need for this new type of thinking. Most children have seen members of their families read books and papers, so that the youngsters come to school with a desire to learn to read. Far fewer of them, however, see number work in the home, so that there is little help from this source in developing an awareness of need.

Another aspect of thinking in arithmetic which proves confusing to some children is the exactness that is required. During his earlier years, the child has recognized that he is older or younger than certain other children, and "larger" and "smaller" have become meaningful terms. But the degree of exactness required of him when he starts to count out eight pencils—no more and no less—is new and bewildering. Development of this type of thinking is a slow and frequently laborious process.

It is hoped, of course, that as skills in quantitative thinking are developed, they will carry over into other areas of study. In order to encourage this, many teachers make an effort to coordinate the study of arithmetic with the work in science, social studies, language arts, or other subjects.

Adults accept the fact that the type of thinking learned in arithmetic is valuable throughout life. This is not very real to smaller children. At the upper elementary levels, however, many students respond well to real-life problem situations at the adult level. Hence many textbooks include problems on taxes, installment buying, or payment of interest on a loan.

C. The Learning of Arithmetic

It would make the lives of teachers and students much easier if they could assume that "once learned, always remembered." Learning is in effect a growth process, however, and growth is generally slow. Consequently, learning arithmetic involves repeated contacts with the same material. To illustrate: Who can say how many times he learned, forgot, and relearned the process of long division?

LEARNING A CUMULATIVE PROCESS

A growing body of knowledge about the nature of learning has influenced the arithmetic curriculum in numerous ways. Consider again the learning of the multiplication facts. A few generations ago, these were presented in a table, going through 12 twelves. As was mentioned earlier, some books presented this table early in the year, and students didn't progress beyond that point until they knew the table "by heart." These same facts are now presented in groups or families, and some study of them occurs in each of several grades.

Another illustration of this kind of presentation is found in the study of addition. Although newer texts tend to place the study of some addition facts and topics at lower grade levels, several textbook series have followed this pattern: *Grade 1*, study addition facts with sums through 6; *Grade 2*, study addition facts through sums of 12, column addition, and adding two-place numbers; *Grade 3*, study the remaining addition facts, along with carrying and higher-decade addition. Although the time spent studying addition decreases as the student moves through the grades, some time is used in the study of this operation during each year of elementary arithmetic.

One result of the cumulative approach is that teachers are able to build toward the more difficult operations. Recent inquiries about the level of difficulty of the operations in arithmetic help us to decide at what grade level the operations should be taught. Because of the cumulative learning processes used in arithmetic, however, a student who fails to grasp a principle in a lower grade, will be handicapped in related work at subsequent grade levels. Thus, a student who fails to comprehend the concept of borrowing at third-grade level may still be making errors in borrowing at sixth- or seventh-grade level. For this reason, arithmetic teachers need to develop a diagnostic approach to much of their work.

PSYCHOLOGY OF ARITHMETIC

It would serve no useful purpose to include a detailed description of the psychology of arithmetic in this book, but certain general principles will be mentioned.

The drill theory. The National Society for the Study of Education Yearbook for 1930[8] presents an extensive treatment of the teaching of arithmetic from the stimulus-response point of view. This view had been presented by Thorndike and his associates, initially as laws, but later as descriptions of the learning process. Generally, this approach to learning as applied to arithmetic might be described thus: (1) a stimulus tends to produce a response; (2) if the response produces satisfaction, there will be a tendency to form a bond, so that further recurrences of the stimulus will produce the same response; (3) the bonds are strengthened through use and weakened through disuse.

This type of thinking dominated arithmetic teaching for several decades. Suppose the student was trying to learn that $8 \times 6 = 48$. The teacher would drill him at great length to try to get the stimulus (in this case 8×6) to yield the proper response (48). Correct responses would yield dividends in personal satisfaction and in teacher commendation, both of which would be absent if the response was incorrect. Then the bond, having been established, would be strengthened by extensive usage. In essence, this whole theory as applied to arithmetic led to an extreme emphasis on drill.

[8] "Some Aspects of Modern Thought on Arithmetic," Part I, *Twenty-Ninth Yearbook*, National Society for the Study of Education.

have been published. Out of these and related research projects has grown the realization that the best learning is that which is based upon real, meaningful situations.

One result of the meaningful teaching approach has been the realization that maturation is of major importance in the teaching of arithmetic. One of Thorndike's descriptions of learning was that, when a person is ready to act, action gives pleasure, whereas inaction results in annoyance. Conversely, if a person who is not ready to act is forced to act, annoyance is the result. As applied to arithmetic, if a child is forced to operate at a level which has no meaning for him, the learning that results will be transitory.

As previously mentioned, studies of the learning process have shown that, for many decades, children were being taught material in arithmetic which was both meaningless and so difficult that few students were experiencing success in it. As a result, many topics formerly taught in the third grade have been moved to fourth or even fifth grade.

Maturation is also a factor in determining the rate at which a student progresses from the concrete to the abstract. Moving too fast gives a student a sense of frustration because he is not ready for abstract work; moving too slow is equally annoying, since the use of concrete objects by a student ready for abstract work produces a feeling of lack of progress.

The emphasis on reteaching is another change which has occurred because of increased information about how children learn. Since few people actually learn material as a result of a single contact, arithmetic programs are designed to have children work with the same material several times and at a variety of grade levels. For example, students who study $3 + 3 = 6$ in first grade study the very same addition fact in several subsequent grades. With each new treatment, the fact is given less time and emphasis, but it is retaught repeatedly. Meaningful and fairly permanent learning usually results from such reteaching.

It should be pointed out that reteaching does not necessarily mean review. In many classrooms, *review* has come to mean a hurried coverage of topics or material completed earlier. Reteaching, however, implies a different operation designed to maintain or extend concepts and skills on topics previously studied. Such teaching would be like that done on a new topic, except that it is done at an accelerated pace.

Meaningful learning may grow out of practical applications. Hence, many teachers manage to use arithmetic in science, social studies, or playground activities. The best type of arithmetic learning results when

structure of the Hindu-Arabic decimal system of numerals and the basic computational processes to elementary algebra and geometry and the nature of logic. They point out, however, that a traditional college course in these topics designed to develop mechanical skills would be unsatisfactory. The elementary school teacher needs to know how to develop principles and the understanding of concepts. These writers also recommend that teachers of arithmetic in grades seven and eight minor in mathematics.

A common error in setting up mathematics courses for elementary teachers is to give much attention to content but to assign the classes to professors who lack the proper perspective. Such courses as taught by graduate students, by mathematicians who feel that their status has been compromised, or by people who have no knowledge of, or interest in, the problems of an elementary teacher would be of doubtful value.

Methods. At the opposite extreme from the subject-matter specialist is the methods advocate. Through such catch phrases as "we teach children, not arithmetic," he tries to convince himself that skillful teaching is all-sufficient. Incidentally, his tribe is not numerous on the current educational scene. It is usually accepted as fact that, regardless of skill in teaching, one has little chance for success in trying to teach what he does not know. Hence, knowledge of teaching methods alone would not ordinarily be considered as adequate training for the elementary teacher.

SUBJECT MATTER PLUS METHOD

Out of the conflict just described, various compromises have been reached. One is that an elementary teacher needs a certain amount of college training in mathematics but this training should be of a nature that will contribute to the teacher's arithmetic background. Another is that teaching arithmetic requires certain types of skills that are not normally required in teaching other subjects. Hence, some work in these specialized methods is important to the teacher.

Although practices in this area vary widely from one state to another or from one college to another, a fairly typical program might be to give elementary trainees a year of college mathematics (six semester hours) along with a two- or three-hour course in arithmetic methods. The mathematics advocates say more mathematics should be required; the methods advocates take the opposite position. But this seems to be a fairly effective compromise.

Presumably no one could claim that final answers have been reached in the training of arithmetic teachers. It is important that elementary teachers, administrators, and college personnel constantly seek new and better ways of providing such training. It is hoped that we can refrain from going overboard for new ideas simply because they are new—or discarding older practices just because they are old.

One problem that deserves careful attention is the matter of teacher attitude toward arithmetic. Recently, one of the authors made an informal survey (presently unpublished) among a group of college juniors and seniors, all of whom were planning to be elementary teachers. This survey found that less than 10 per cent of the group liked arithmetic, whereas more than one-half of them expressed an active dislike for it. Such attitudes are likely to be taught to children, since teachers find it very difficult to conceal their true feelings in such matters.

Something to Think About

1. How would you describe your own attitude toward mathematics?
2. What person or experience was particularly instrumental in forming your attitude?
3. In your opinion, is it possible for a teacher who dislikes mathematics to teach students to like it? Explain.
4. To what degree does mental discipline function in a modern mathematics program?
5. Who decides as to the proper grade placement of a topic in arithmetic? Upon what basis is such a decision made?
6. What is your position as to the proper balance between content and methods in training elementary teachers? Explain.
7. On the basis of your readings in current literature, what do you consider to be the more promising developments in the teaching of arithmetic? Prepare a class report on one of them.
8. Are Federal funds being used in the teaching of arithmetic? Explain.
9. Could you justify devoting more time or money to the teaching of arithmetic than to teaching in other subject areas? Explain.
10. Many teachers say that boys do better work in arithmetic than girls do. Can you find any research that would support this position?

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The Development of Number Concepts and Numeration

Rapid developments in science and technology have focused attention on mathematics. The general public has been made aware of the important role mathematics plays in our society. It is likely to play an even more important role in our future. As a result, searching examinations of the teaching of elementary school arithmetic and high school mathematics are being made.

A textbook on methods of teaching arithmetic presupposes some basic knowledge of number and number operations. Sufficient knowledge may or may not have been acquired by the prospective or present elementary school teacher. Since there has been a definite trend away from teaching arithmetic as a series of mechanical operations to teaching arithmetic as a logically developed structure, teachers must understand more about number, numerals, and operations than ever before. A portion of this textbook is devoted to information that should help the teacher appreciate the systematic concept of our number system and the role it has played in human development.

This chapter presents the following topics:

A. What Is Mathematics?
B. Numbers and Numerals
C. Counting
D. Systems of Numeration
E. Sets and Numbers

A. What is Mathematics?

What is mathematics? Is it various operations we do with numbers? Is it a body of "truths" that have been "discovered?" Maybe it is just man's way of describing the world about him. But what about the abstract flights of fantasy taken in some branches of higher mathematics? Are these related to "reality"? Efforts to give a concise, meaningful description of mathematics are unavailing. Court[1] observed that mathematics has two aspects. It furnishes us with a description of the world we "see" around us and tools for describing other parts of the world. This may be described as the functional side of mathematics. Then there is the philosophical side of mathematics. This side is concerned with study of the foundations of mathematics and the extent of the validity of mathematical processes. The philosophical side of mathematics is of little interest to non-mathematicians and even perhaps, to only a few mathematicians. Yet such considerations have profoundly influenced the development of mathematics in the past century.

Some people regard mathematics as a game involving symbols manipulated according to man-made rules. The game consists of making more definitions and rules and investigating their application to the symbols used in the game. Other people believe that we have an intuitive sense of number. Certain basic understandings about quantity and number may not need defining. We are simply born with these understandings. On them we may build more complex concepts by definitions and rules. In this chapter, we shall present a brief intuitive concept of number and then a more abstract or formal way of defining number and number operations.

Mathematics is abstract. Mathematics in the elementary grades must be focused on applications of number and the manipulative techniques which are a part of mathematical knowledge. The teacher, however, and, as soon as possible, the child, must realize the abstract nature of mathematics. The child must learn to abstract the concept of threeness from three objects. Only in this way can children build up the patterns of generalizations necessary to proceed to more complex concepts of number.

Mathematics is reasoning. We learn about things in two ways. One way involves observing, collecting facts or data, and experimenting.

[1] Nathan Altshiller Court, *Mathematics in Fun and in Earnest* (New York: The Dial Press, Inc., 1958), p. 170

This is the way the scientist approaches a problem. From the gathered data, the scientist attempts to formulate a general rule or principle which describes the situation from which the data were taken. This is called the *inductive* method of reasoning.

Abstract reasoning is another method. Certain propositions may be accepted as true. Working with these assumptions, then certain other propositions may be proved. This is called *deductive* reasoning. Mathematics is usually considered to be the science of deductive reasoning However, the mathematician and the scientist use both of these methods of reasoning in their experimentation and problem solving.

Mathematics is systematic. A number system is not simply a collection of varied and sundry ideas, operations, definitions, and "facts." Instead, a number system is a single, logically constructed structure with many complex and interrelated parts. These parts may be studied or examined separately, but out of this examination should come some understanding of the whole. Examinations of number systems have shown that we have not one number system, but many. An understanding of the relationships between number systems is essential to an understanding of mathematics.

B. Numbers and Numerals

What is number? What are numbers? Are numbers things you use in counting? If so, then what are $\frac{1}{2}$, $\frac{2}{3}$, or any of the other fractions? Are they numbers? Is a number the mark you write on the chalkboard, or is it simply a symbol for something? Are number operations something you do with numbers, or are they something you do *with symbols*?

These questions indicate something of the kinds of problems which concern those engaged in improving the teaching of arithmetic. Some of the current difficulties in arithmetic teaching may stem from a lack of understanding of these seemingly simple questions. The elementary school teacher may or may not be comforted to know that great mathematicians and philosophers have pondered the foregoing questions and have not reached agreement on the answers.

BASIC CONCEPTS OF NUMBERS

Several points of view have been adopted by those interested in basic foundations of numbers. A summary of some of these points of view will aid the teacher in understanding "modern mathematics."

Intuitionism. Those holding to this concept of mathematics emphasize intuition, experience, and immediate insight as the source of understanding of number. Our intuition forces us to recognize certain concepts related to quantity or plurality and to draw certain conclusions which are clear and undisputable. The intuitionists call many number concepts *self-evident.* Related to these concepts are those which hold that basic number concepts are a part of our being once and for all. Number concepts are said to exist a priori, that is, apart from human reasoning or experience. In other words, we come to understand number because the universe just exists that way.

Formalism. Others believe that we may reduce number to a few deep-lying propositions which we know by intuition or accept without proof. Then, all other concepts of number may be deduced from these undefined accepted propositions. This is also called the *axiomatic approach* to number. The undefined propositions are called *axioms.* One of the early leaders of this school of thought was Peano who reduced number to five basic propositions in an effort to determine the very least number of propositions from which the remainder of mathematics may be deduced. Formalism holds that, given a few accepted axioms, signs, and rules all of mathematics may be reasoned out in a mechanical manner. The formalists place emphasis on consistency in using the systems of symbols and rules which they define.

Logicism. Starting with what seem to be the simplest and most plausible elements, the logicists attempt to construct completely the basic ideas in mathematics. They do not wish to tie number concepts to reality, that is, the world of experience. Instead, their goal seems to be setting up a system of symbols and rules and then exploring all the resulting relationships that may be deduced from these. Mathematics becomes a game played according to the rules the player makes.

Early logicists used sets and classes in their efforts to show the equivalence of the "body" of mathematics and systems of logic. Several criticisms have been made of the logical approach to explaining number. There may be some doubt whether some of the necessary concepts associated with sets, such as one-to-one correspondence and successor, are more basic than the concept of cardinal number or counting itself. Also, it has been shown that systems of logic are simply tautological relations or relations already inherent in the definitions and linguistic rules of the systems.[3] Therefore, a mathematics based on symbolic

[3] Friedrich Waismann, *Introduction to Mathematical Thinking* (New York: Harper and Row, Publishers, 1959), pp. 70–71.

logic seems to be a huge system of tautologies and its propositions—truths without content. Subsequent efforts by Russell and others have satisfied some of the critics of the logicist's approach to number.

Modern ideas of number. By restricting the use of such words as "all" and "infinite," the theories of sets or classes offer possibilities of a systematic development of number concepts. Our language plays a part in how we form concepts, including number concepts. Suppose we return to our question, "What are numbers?" How do we explain any abstract concept to a person? We would show him various examples of how the word representing the abstract concept is used. Consider the question, "What is beauty?" We would cite as examples: "She is a beauty"; "Its beauty has faded"; "Beauty is only skin deep"; "He appreciates beauty"; "True beauty comes from within." We would combine this word with other words, use different syntactical patterns, explore all the ways our language permits us to use the word. The person who can use the word in all these ways understands it. No kind of "definition" can give him a better understanding.

The question, "What are numbers?" may be approached from this viewpoint. We simply illustrate in many ways the uses of the word "number" and the numerals. Out of this usage comes the understanding we seek. Obviously, the instances of number which may be demonstrated are limited. We may demonstrate some intuitively evident group of things and say "three chairs," or "four erasers." For large groups, however, this demonstration of an instance of number breaks down. We may use numerals in examples to illustrate their use in counting. By these means we develop certain elementary concepts of number and number symbols.

If we developed only these elementary concepts and usages, most of mathematics would be forever beyond our grasp. We develop certain concepts that are compatible with reality or experience, and from these the necessary rules of arithmetic follow. No rules of logic can be more basic than the manner by which we first learn about numbers. The more advanced propositions of arithmetic, however, should be demonstrated as logically derived truths based on the elementary concepts.

INTUITIVE CONCEPTS OF NUMBERS

The numbers 1, 2, 3, 4, . . . are called *natural numbers.* This name, no doubt, arose because of an intuitive feeling that the counting numbers had an existence apart from the mind of man. We also use fractions

such as $\frac{1}{2}$ and $\frac{2}{3}$. Now, fractions do not arise through any process of counting, but arise when measurements are made. Obviously, natural numbers and fractions are two kinds of numbers, yet we equate natural numbers and fractions. For example, we say that the fraction $\frac{6}{2}$ equals the natural number 3. This is written in what is called an *equation*. For example,

$$\frac{6}{2} = 3$$

With our intuitive concept of fractions and natural numbers, we understand this. But natural numbers and fractions may be developed in a strictly logical manner unrelated to intuitive feelings for quantities or parts of wholes.

Still a different kind of number arises as a result of measurement when we consider a number such as the square root of 2. Usually, this number is written by the symbol "$\sqrt{2}$." A rather famous proof, to be demonstrated later, has shown that the $\sqrt{2}$ cannot be equated to either a natural number or a fraction. Such numbers are called *irrational numbers.*

An example of another kind of number is π. This number is the ratio of the circumference to the diameter of a circle. Measurement again gave rise to the need for a definition of this number. It may be shown that π cannot be equated to any natural number or fraction, neither may it be equated to any root of a natural number or fraction. Such numbers are called *transcendental numbers.* The number π plays such an important role in modern mathematics that it would have been discovered entirely apart from the circle.

All these kinds of numbers arose intuitively through efforts to count objects or make physical measurements. For many centuries, the intuitive concepts were the only ones. Gradually, man has built a logical structure to tie together all these kinds of numbers. By making a few basic premises, all of these kinds of numbers may be logically derived from the natural or counting numbers without reference to intuition or experience.

NUMBER SYMBOLS AND NUMERALS

There is evidence that many children and adults confuse numbers with their symbols. Seldom, if ever, is there any occasion in the use of

language to confuse an object with its name. Fouch and Nichols[3] have identified several ways in which people confuse numbers and the names of numbers. These range from confusing the size of numbers with the physical size of the written symbols representing them, to confusing number operations with the processing of numerals or number names.

What are number symbols? There are ten symbols used in the decimal system of numeration. These are listed below:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

The name of a number is a *numeral.* Numerals are composed of the symbols just given. Also, the name may be a word. The symbol "3" is distinguished from the number 3 in written discourse by enclosing it in quotation marks. In a similar fashion, we distinguish between a word and its referent.

Much emphasis has been placed on accurate language in connection with modern mathematics. Beberman[4] emphasized the need for precision of language and pointed out that the University of Illinois Committee on School Mathematics has given careful attention to the problem of helping children distinguish between symbols and referent.

It should be realized that a number may be written many ways. Listed below are several names for the number 5:

five $\quad 3 + 2 \quad 9 - 4 \quad \sqrt{25} \quad 125^{1/3}$

$\frac{65}{13}$ $\quad$ v $\quad$ cinque $\quad$ fünf

In the same way, a fraction may have several names. Some of the names for the fraction $\frac{7}{2}$ are

seven halves $\quad 7 \div 2 \quad 3\frac{1}{2} \quad \frac{14}{4}$

$$\sqrt{\tfrac{49}{4}} \qquad \tfrac{10}{2} - \tfrac{3}{2} \qquad \tfrac{5}{2} + 1$$

Although the foregoing symbols all refer to the same number, namely, the fraction $\frac{7}{2}$, note that quite different symbols are used. Changing symbols to a standard symbol referring to the same number sometimes involves operations on numbers.

[3] Robert S. Fouch and Eugene D. Nichols, "Language and Symbolism in Mathematics," *The Growth of Mathematical Ideas,* Twenty-Fourth Yearbook National Council of Teachers of Mathematics (Washington, D.C.: The Council, 1959), pp. 327–369.

[4] Max Beberman, *An Emerging Program of Secondary School Mathematics* (Cambridge: Harvard University Press, 1958), pp. 4–23.

Some writers have advocated very precise language to avoid confusing numbers and numerals.[5] Others, while recognizing the need for precision, seem to feel that insistence on this distinction in writing or speaking may become somewhat tedious.[6] If the reader is likely to be uncertain about which is meant, the number or the numeral, perhaps the distinction should be made clear. Most writers apparently still use the word "number" to refer to either a number or the symbol for it. That practice has been followed in this text with occasional reminders where needed.

NUMBER AND NUMBER SYSTEMS

We should understand the difference between a number system and a numeral system. By a *numeral system* we simply mean a set of symbols and the plan by which they are used to designate a number. Thus we have the Roman numerals, the Hindu-Arabic numerals, and other systems for designating numbers with symbols. Two important concepts in our system of numeration are *base* and *place*.

Our concepts of the natural or counting numbers grow out of our experiences with objects or things. Therefore, we come to understand what is common to a pair of shoes, a pair of socks, and a pair of feet. This common property we call *two* and denote by a symbol "2." Note that this common property is independent of the name or symbol which we attach to it. Thus we come to form an intuitive idea of the natural or counting numbers. These numbers and some "operations" on these numbers make up what we call a *number system*.

Addition of numbers. With an intuitive idea of the natural or counting numbers, some ideas about the operation of addition may be formed. With any set of objects there may be associated a natural number. If two sets of objects are put together to form a single set, there is a natural number associated with the united set. This natural number is called the *sum* of the first two numbers. Addition then is called a *binary* operation. For any pair of natural numbers there is associated another natural number called the *sum*. The operation of association is called *addition*. The usual notation for this operation is as follows:

$$4 + 5 = 9$$

[5] *Ibid.*

[6] *Report of the Commission on Mathematics, Appendices* (New York: College Entrance Examination Board, 1959), p. 4

Note that the order in which the "4" and "5" are written does not matter. The rule may be stated this way: If a and b are any pair of natural numbers, then:

$$a + b = b + a$$

This property is called the *commutative law of addition.*

This binary operation may be extended to three numbers. The three numbers 2, 4, and 5 may be added by adding the first two numbers and then adding the sum to the third number. If the additions are carried out, it may be seen that the pair selected to be added first does not affect the final sum. For example:

$$(2 + 4) + 5 = 6 + 5 = 11$$

Also, $$2 + (4 + 5) = 2 + 9 = 11$$

The rule may be stated as follows: If a, b, and c are any three natural numbers, then:

$$(a + b) + c = a + (b + c)$$

This is called the *associative law of addition.*

Multiplication of numbers. An intuitive idea of multiplication may also be formed. If a and b are any pair of natural numbers, then to multiply a by b set up a rectangular array of objects. Let there be a objects in each row and b objects in each column. The natural number associated with all the objects in this rectangular array is called the *product* of a and b. This is written $b \times a$ and is read "b times a."

For example, 3 times 4 may be set up as follows:

```
*   *   *   *
*   *   *   *
*   *   *   *
```

The natural number associated with this set is 12. Therefore,

$$3 \times 4 = 12$$

It is easily seen that 4 times 3 is also 12. Therefore, order does not matter in multiplication of natural numbers. The rule may be stated this way: If a and b are natural numbers, then

$$a \times b = b \times a$$

This is called the *commutative law of multiplication.*

Multiplication of natural numbers, like addition, obeys the associative law. This means that:

$$(a \times b) \times c = a \times (b \times c)$$

The operations of addition and multiplication combine to give a fifth law. This law may be illustrated by considering the array of objects by which multiplication was intuitively defined. The array below illustrates 4 times 8.

```
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
*   *   *   *   *   *   *   *
```

This would be written 4×8. Suppose this array of objects is separated into two groups. The union or addition of the two groups would

```
*   *   *       *   *   *   *   *
*   *   *       *   *   *   *   *
*   *   *       *   *   *   *   *
*   *   *       *   *   *   *   *
```

equal the original group by our intuitive definition of addition. The first group may be thought of as 4×3, however, and the second group as 4×5. Therefore,

$$4 \times 8 = (4 \times 3) + (4 \times 5)$$

Also, each row of 8 objects may be thought of as 3 objects added to 5 objects. That is,

$$8 = 3 + 5$$

Therefore, the total number of objects in the array may be thought of as

$$4 \times 8 = 4 \times (3 + 5)$$

Now, since 4×8 has been written two different ways, it is correct to equate these two different symbols for the same thing.

$$4 \times (3 + 5) = (4 \times 3) + (4 \times 5)$$

In general, if a, b, and c are natural numbers, then

$$a \times (b + c) = (a \times b) + (a \times c)$$

This is called the *distributive law* and indicates that multiplication is distributive with respect to addition.

Frequently, in a series involving multiplication and addition, the parentheses are left out. It is understood that the multiplications are done before the additions.

Number systems. Intuitive ideas of the natural or counting numbers and two binary operations have been presented. It was found that these operations on natural numbers obeyed five laws. At this point, one may begin to make generalizations about the meaning of number. The intuitive idea of number as a counting device may begin to give way to an abstract concept of number as simply an entity that obeys certain rules. Any set of objects or entities for which the binary operations of addition and multiplication may be defined, and for which these two operations obey the five laws, may be called *numbers*.

A *number system* may be defined as any set of objects or entities on which two binary operations called *addition* and *multiplication* are defined, such that both operations are commutative and associative and multiplication is distributive with respect to addition.

With this definition there is not just one number system, but many. Number systems become more than just an intuitive idea of quantity. Modern mathematics is concerned with the logical development of number systems and number operations, given a few basic assumptions or definitions. It will be shown later that the natural number system is a part of a larger number system which is part of a still larger system, and so on. In each of these systems there are defined two binary operations that obey the following laws:

1. $a + b = b + a$	commutative law of addition
2. $(a + b) + c = a + (b + c)$	associative law of addition
3. $a \times b = b \times a$	commutative law of multiplication
4. $(a \times b) \times c = a \times (b \times c)$	associative law of multiplication
5. $a \times (b + c) = (a \times b) + (a \times c)$	distributive law of multiplication with respect to addition

C. Counting

Basic to all concepts of number is counting. Even the most primitive tribes have shown some concepts of number. The evolution of counting from the most primitive cultures to the present has paralleled the development of civilization.

PRIMITIVE COUNTING

The origin of number has caused much discussion and some controversy in connection with the study of arithmetic. Although there has been much research on the use of number in primitive cultures, the origin of the concept of number and the process of counting remains largely unknown.

The evolution of man into tribes or groups required at least simple counting. The chief needed to know how many tribesmen he had or how many animals he possessed. Probably the earliest method of counting was by using some kind of tally. In counting sheep or warriors, a one-to-one correspondence would be kept with fingers, pebbles, sticks, or marks in dirt or on a stone.

Many primitive tribes have been found that apparently had concepts only of 1 and 2, or 1, 2, 3, and many. They would develop a few words to express small numbers after which the word "many" was used to indicate numbers beyond their comprehension. Other studies show that many kinds of animals possess a primitive number sense or counting sense.

NUMBER BASES

Properties of the decimal system of numeration become overlooked because of familiarity with the system. Often, a consideration of numeral systems with bases other than 10 will aid in understanding the characteristics of the decimal numeral system.

It is known that certain African natives used 2 as a base. Their counting went "1, 2, 2 and 1, 2 and 2, much." Any quantity above 4 was referred to as *much*. Other tribes are reported to have no number name beyond 3, indicating that 3 would be used as a base for further counting.

The first widely used base or scale was the quinary. The five fingers of the hand formed a convenient matching device for counting or tallying objects. Some South American tribes have been observed to count by hands, that is, "1, 2, 3, 4, hand, hand and 1, hand and 2," and so on. Even now numbers less than 10 are called *digits*.

Very probably, when the fingers of one hand came to be used in counting, the fingers of the other hand were soon used. It is only natural that the base 10 would come into use. No one knows the exact details of how the Hindu-Arabic numeral system, using a base of 10, developed and was later spread to most of the world. It is thought

that the symbols for the nine digits and zero, much as they are written today, were in use before the eleventh century.

The duodecimal system seems to have been used in prehistoric times. It still exists in many of our measurements. The number of inches in a foot, ounces in the ancient pound, units in a dozen, hours around the clock, and months in the year are examples. This base may be divided into several integral fractional parts. The Duodecimal Society of America disseminates information and literature on the advantages of the dozen base. Publications of this group have pointed out many advantages in using a duodecimal system of numbers.

A scale of 20, the vigesimal, was also used in prehistoric times. Languages of such widely separated peoples as the Aztecs of Mexico and the ancient Celts show traces of the 20 base system of numerals. The Aztecs had an elaborate system of numerals using 20 as a base. Present French use of *quatrevingt* (4-20) for 80 and similarly derived expressions are evidence of the use of 20 as a number base.

Babylonians employed the number 60 as a sort of base. The sexagesimal base still exists in the division of degrees, hours, and minutes into 60 parts. It is probable that 60 came into use because it has so many integral divisors (2, 3, 4, 5, 6, 10, 12, 15, 20, and 30) which made working with fractional parts easier.

MODERN CONCEPTS OF COUNTING

Although the decimal system of numeration is now in use almost throughout the world, numeral systems with other bases are being used. Modern computers are employing the binary numeral system because it uses only the symbols "0" and "1." These can easily be made to correspond to "on" and "off" for certain kinds of electronic circuits. These circuits can then be stacked together in large quantities to perform operations on numbers expressed in a binary system of numerals. Conversion from one base to another is needed in these operations.

Reasons for counting. Counting may be done for two reasons: to answer the question, "How many?" and to answer the question, "Which one?" In the first case, where quantity is involved, the number is said to be a *cardinal* number. In the second case, where order or position in a series is involved, the number is said to be an *ordinal* number. These different uses carry over into the number names. The ordinal use of "first," "second," "third," and so on, is easily

recognized as indicating order or position. The number names "one," "two," "three," and so on, may in common usage represent quantity or order and, therefore, refer to cardinal or ordinal use of number.

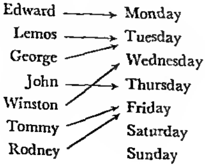
Just as in the days of primitive man and the tally, the idea of one-to-one correspondence is basic to counting. Objects being counted are placed in a one-to-one correspondence with the set of natural numbers. This is done by pairing an object and a number name. The set of number names is ordered. The last number name paired with an object in the group of objects being counted is the cardinal number of the group.

Mapping. A more modern concept is the idea of "mapping." A *mapping* is a matching operation between two sets of objects. (A collection of objects, of any thing whatsoever, is commonly referred to as a *set*, or a *class*.) Suppose in December you want to know how many payments are left on your car note. The last payment is in April. You might call off the names of the months, January, February, March, and April, and, for each month, turn down a finger. The mapping would be as follows:

January ⟶ little finger
February ⟶ ring finger
March ⟶ middle finger
April ⟶ index finger

Since the thumb remains, you would conclude that you had four payments to make. Notice that the mapping proceeds in one direction. Now, if the set of fingers turned down is considered, then double-headed arrows may be used to show that, for each of these fingers, there is a month. In this case, the two sets are said to be in a one-to-one correspondence.

Now consider a set of names of people mapped into the set of names of days on which they were born.



This mapping differs from the other one. Two names are mapped into Tuesday and into Friday. This is a *many-to-one* mapping.

Bertrand Russell has observed that, in Christian countries, the relation of husband to wife is one-to-one; in Mohammedan countries, it is one-to-many; and in Tibet, it is many-to-one. The accuracy of the first-named example may be open to doubt, but this illustrates the concept.

Two sets are in a one-to-one correspondence when there is a reversible mapping that assigns to each member of one set one, and only one, member of the other set. When two sets can be put into such a correspondence by some mapping, they are said to *contain* the same number of objects or have the same cardinal number.

It may readily be seen that associated with each cardinal number there is a collection of sets. All sets consisting of pairs of objects are associated with the number "two." Sets consisting of only a single object are associated with the number "one," and so on. All sets will belong to one of these collections of sets. Determining how many objects are in a set is the same thing as determining the collection to which the set belongs. Russell used this basic idea to define number.

Counting is done by matching objects with number names. We count four objects by saying, "one, two, three, four." We have then placed the four objects in a one-to-one correspondence with a set (of natural numbers) having the cardinal number "four." Therefore, we say we have "four" objects. This has been a summary of counting and sets that has appealed rather strongly to our intuitive sense of number. A more logically derived "definition" or concept of the natural numbers will be presented later in this chapter.

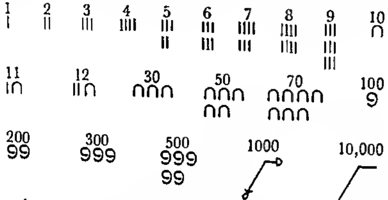
D. Systems of Numeration

As man and his culture developed, the need for some kind of permanent device for recording numbers grew. Early uses of tallies, pebbles, or notches were attempts at recording quantities. People probably used speech to communicate thousands of years before they learned to write words. Likewise, people used number names long before they devised signs or symbols for numbers. The word "three" was in use long before the numeral "3." In the discussion that follows, the term *number system* will frequently mean "system of numeration." This is in accord with common usage.

ADDITIVE SYSTEMS

The simplest sort of written number system involved the additive principle. Different symbols would be used for 1, 10, 100, and higher powers of 10. The symbols would simply be repeated the necessary number of times. Where bases other than 10 were used, symbols for 20, 50, 60, or other numbers would be employed.

More than five thousand years ago, the Egyptians used such a number system. The hieroglyphics which they used have been found inscribed on stone. A simple version of their numerals is as follows:



Any number may be expressed by using the symbols additively, repeating each symbol the necessary number of times. For example,

This represents the number 23,567.

The best-known example of a simple additive or grouping system is the familiar Roman numeral system. The seven most common symbols used to write Roman numerals are as follows:

I	V	X	L	C	D	M
1	5	10	50	100	500	1000

The Romans used symbols for powers of 10: 1, 10, 100, and 1000, and for the numbers 5, 50, and 500. The subtractive principle is also used in forming certain numbers, such as IX for 9, or 1 less than 10; XL for 40 and XC for 90, or 10 less than 50 and 10 less than 100. The additive principle may be seen in such numbers as VI for 6 and XI for 11, meaning one more than five and one more than 10; and LX for 60 and CX for 110, meaning 10 more than 50 and 10 more than 100. The subtractive principle was used very little in ancient and medieval times.

In ancient times, a number might have been written as follows:

$$1964 = \text{MDCCCCLXIIII}$$

Using the subtractive principle, the number would be

$$1964 = \text{MCMLXIV}$$

Computation using Roman numerals, in comparison with the Hindu-Arabic decimal system, must be considered rather inconvenient. Nevertheless, although more modern systems of numerals were generally known in Europe in the eleventh century, Roman numerals were used in bookkeeping until the eighteenth century, partly because it is relatively easy to add and subtract Roman numerals written according to the additive principle. This is shown in the accompanying example:

Addition	*Subtraction*
DCCC L X VIII (868)	DCCC L X VIII (868)
CC X VII (217)	CC X VII (217)
M L XX XV (1085)	DC L I (651)

Multiplication and division using Roman numerals are much more difficult than with modern numerals. Usually, some device, such as a counting board, was needed to multiply numbers.

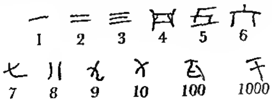
MULTIPLICATIVE SYSTEMS

Some early systems developed into what is called a *multiplicative grouping system*. If the base 10 was selected, then there would be symbols for 10, 100, 1000, and so on. Also, there would be symbols for 1, 2, . . . , 9. These two sets of symbols would be used with multiplication to show how many units of the larger symbols are used. For example,

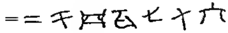
$$2476 = 2(1000) \quad 4(100) \quad 7(10) \quad 6$$

where the proper symbols would be employed on the right side.

An example of such a system is the Chinese, later adopted by the Japanese. The system employed symbols similar to the following:



Example:

2476 = 

CIPHERED SYSTEMS

In a ciphered numeral system with base 10, symbols are adopted for 1, 2, . . . , 9, 10, and for multiples of 10, 100, 1000, and so on. Although such a system uses many different symbols, the representation of numbers is short.

Early Greeks had a method of writing numbers by making use of the initial letters of the number names. These were combined in a simple additive system. The Ionic, or alphabetic, Greek numeral system employed the letters of the Greek alphabet. Letters and the numbers they represented are as follows:

1	α	alpha	8	η	eta	60	ξ	xi	400	υ	upsilon
2	β	beta	9	θ	theta	70	o	omicron	500	ϕ	phi
3	γ	gamma	10	ι	iota	80	π	pi	600	χ	chi
4	δ	delta	20	κ	kappa	90	ϙ	koppa	700	ψ	psi
5	ε	epsilon	30	λ	lambda	100	ρ	rho	800	ω	omega
6	ϝ	vou	40	μ	mu	200	σ	sigma	900	ϡ	sampi
7	ζ	zeta	50	ν	nu	300	τ	tau			

Bars or accents were used for larger numbers, as in the Roman numeral system. Examples of numbers in the Greek system are

$$18 = \iota\eta \qquad 84 = \pi\delta \qquad 643 = \chi\mu\gamma$$

POSITIONAL SYSTEMS

The Hindu-Arabic system is a positional numeral system with base 10. Thus, there are 10 basic symbols: 0, 1, 2, . . . , 9. Now any number may be represented in the form

$$a_n(10^n) + a_{n-1}(10^{n-1}) + \cdots + a_2(10^2) + a_1(10) + a_0$$

(10^2 means 10×10, 10^3 means $10 \times 10 \times 10$; in general, 10^n means the number obtained by using 10 as a factor n times.) Where $0 \leq a_i \leq 9$, $i = 0, 1, \ldots, n$. For example, the numeral 7862 means

$$7(10^3) + 8(10^2) + 6(10) + 2.$$

Therefore, it is seen that the 7 in the numeral stands for 7×1000 or 7000. The 8 stands for 8×100 or 800, and the 6 stands for 6×10 or 60. The numeral 7862 means

$$7000 + 800 + 60 + 2$$

The value of one of the basic symbols depends on the position it occupies. In the numeral 38, the 3 stands for 30. In the numeral 3692, the 3 stands for 3000.

SYSTEMS OF NUMERATION WITH OTHER BASES

Some electronic computers are constructed to perform mathematical operations on numbers expressed with bases other than 10. The binary base, the quinary, and the octonal bases are used. The duodecimal (12) base has been suggested for common use. As mentioned earlier, there is even a Duodecimal Society of America which proposes to conduct research and educate the public in mathematical science with particular relation to the use of base 12 in numeration and mathematics.

The binary base. The only digits in the binary system of numeration are 0 and 1. So simple is this system that even counting is confusing. The first eight binary numerals are

1.	1	5.	101
2.	10	6.	110
3.	11	7.	111
4.	100	8.	1000

Note that the numeral following 111 is 1000. In the system using the base 10, the numeral following 999 is 1000. It may be easily seen that expressing large numbers in the binary system will be a long, tedious process involving many digits. For example,

$$183_{10} = 10110111_2$$

and

$$512_{10} = 1000000000_2$$

Computation using binary numerals and converting from the binary system to the decimal system will be discussed later.

The quinary base. The quinary base system of numeration uses five symbols: 0, 1, 2, 3, 4. The first few numerals in this system are

1.	1	11.	21	21.	41
2.	2	12.	22	22.	42
3.	3	13.	23	23.	43
4.	4	14.	24	24.	44
5.	10	15.	30	25.	100
6.	11	16.	31	26.	101
7.	12	17.	32	27.	102
8.	13	18.	33	28.	103
9.	14	19.	34	29.	104
10.	20	20.	40	30.	110

Note that the numeral 100 follows the numeral 44, whereas in the decimal system the numeral 100 follows the numeral 99. Computation with quinary numerals and converting to the decimal system will be discussed in a later chapter.

The duodecimal base. The duodecimal system contains a dozen digits. The symbol "T" is used for 10 and the symbol "E" for 11. These have been called *dek* and *el*. In this system, it may be seen that the quantity one dozen is expressed by "10" and may be called *do*. The quantity one gross, or a dozen dozens (144), is expressed by "100" and may be called *gro*, for gross.

In counting, we count 1, 2, 3, 4, 5, 6, 7, 8, 9, T, E, 10, or one, two, . . . , nine, dek, el, do. If the counting continues systematically,

Duodecimal System	*Decimal System*
11 = 1 do + 1 unit =	13
12 = 1 do + 2 units =	14
19 = 1 do + 9 units =	21
1T = 1 do + T units =	22
1E = 1 do + E units =	23
20 = 2 do + 0 units =	24
99 = 9 do + 9 units =	117
9T = 9 do + T units =	118
9E = 9 do + E units =	119
T0 = T do + 0 units =	120
EE = E do + E units =	143
100 = 1 gro =	144
1000 = 1 do gro =	1,728

The duodecimal system of numeration offers the advantage of a base with the factors 2, 3, 4, and 6 instead of simply 2 and 5. This simplifies the writing of many fractions in the duodecimal notation. For example,

Fraction	*Decimal*	*Duodecimal*
one-half	.5	.6
one-third	.3333 . .	.4
one-fourth	.25	.3
one-sixth	.1666 . . .	.2
one-eighth	.125	.16
one-ninth	.1111 . . .	.14
one-twelfth	.0833 . . .	.1

Computation with duodecimal numerals and converting to the decimal system will be discussed in a later chapter.

E. Sets and Numbers

The preceding sections have presented a very brief summary of some of the beliefs concerning the basic foundations of number. Philosophers and mathematicians do not agree about the ultimate basis for defining numbers. Some have believed number to be intuitive and an inherent part of man's personality. Others have sought a basic list of axioms from which number may be deduced. More recently, the logical development has been emphasized. By making use of certain simple definitions of sets or classes, a logical development of number may be effected. Whether these definitions are more elementary or basic than the simplest concepts of cardinal or counting numbers may be open to question. For example, in such a development of number, one needs to use such terms as "single" and "pair." Whether these concepts are more basic than the concepts of "one" and "two" may be disputed.

This section presents a simplified, logical development of the cardinal or counting numbers in a way understandable to the non-mathematics major. This presentation should help the arithmetic teacher to understand the modern approach to number. The role played by sets and set operations in laying a foundation for number and number operations should be appreciated.

SETS OR CLASSES

There are many collective words in use in the English language. Some of them have been demonstrated. The term *set* as used in mathematics, however, differs from its everyday use, such as when we say a *set* of dishes, a *set* of toys, or a *set* of teeth.

A *set* will be defined as a collection into a whole of definite, well-defined objects of our thought or perception. These objects are said to be in, or to belong to, the set. They are called *elements* or *members* of the set.

One way to describe a set is to enumerate or list its elements. For example, a family may consist of the father, mother, son, and daughter. George, Tommy, Winston, and John are a set of teachers. An eraser, a piece of chalk, a ruler, and a book make a set, as do the vertices of a square. In all cases, the objects can be perceived or conceived.

Symbolic representation. A set may be symbolized by a capital letter or by listing or writing the elements of the set. For example,

$$A = \{a, b, c, d, e\}$$

or

$$B = \{1, 2, 3, 4, 5\}$$

In listing or writing, the elements of the set are enclosed in braces. The right-hand side shows the elements of sets A and B. We may say that a is a member of A by writing

$$a \in A$$

or that 4 is a member of B by writing

$$4 \in B$$

The symbol $\in$ means "belongs to" or "is a member of." We may denote that f is not a member of A by writing

$$f \notin A$$

A set may be a collection of points in the cartesian plane. It may be a collection of number-pairs. For example,

$$P = \{(5, 0), \quad (5, 4), \quad (1, 4), \quad (1, 0)\}$$

may be a set where each of the ordered number-pairs represents a point. A set may also be made up of sets. For example,

$$A = \{\{1\}, \quad \{1, 2\}, \quad \{1, 2, 3\}\}$$

is a set of sets of numerals.

The null set. A very important set is the empty set. An empty set contains no elements. Consider a bowl of apples. The set of oranges in the bowl is an empty set. One may speak of the number of oranges in the bowl. The number is the empty set. Likewise, the set of boys enrolled in a girls' gym class is an empty set.

Now, it should be noted that the empty set is something. It is a concept. We can talk about it. The empty or null set even has a symbol. A pair of braces, { }, with nothing enclosed represents the null set. It is an important set and will be used later.

Equality of sets. A pair of sets are said to be *equal* when they contain the same elements. When we have $A = B$, read "set A equals set B," then B is just another way of designating the same set of elements designated by A. The order of writing the elements in a set makes no difference. For example, the set $\{a, b\}$ is the same as the set $\{b, a\}$. How many ways can you write the set $\{a, b, c\}$?

Subsets. The set A is called a *subset* of the set B if every element of A is also an element of B. Consider the set $A = \{a, b, c\}$ and the set $B = \{a, b, c, d\}$. Every element of A is also an element of B; therefore, A is a subset of B. A set may be a subset of itself. *Why?*

Consider the set $A = \{a, b, c\}$. The subsets of A are

$$\{a, b, c\}, \quad \{a, b\}, \quad \{a, c\}, \quad \{b, c\}$$
$$\{a\} \quad \{b\} \quad \{c\} \quad \{\ \}$$

There are eight subsets of A. The null set is by definition a subset of all sets. We write the fact that one set is a subset of another as follows:

$$A \subset B$$

For example: $\{a, b\} \subset \{a, b, c\}$. If $A = B$, then A is said to be an *improper* subset of B. The set A is said to be a *proper* subset of B if there is an element of B not in A. *Write some examples of proper and improper subsets.*

The null set may be defined as the subset common to all sets. It is a subset of itself. Now, how do you define a set with a single element in it? We may exhibit many sets with a single element. But we need to start getting away from the concrete and to the abstract if we are to build a number system. A set with a single element is a set whose only proper subset is the null set $\{\ \}$. This is only a tentative start toward defining the natural number 1.

OPERATIONS ON SETS

In order to define formally natural numbers using the "set" or "class" idea, certain operations on sets must be defined.

Union of sets. By *union* of a pair of sets we mean the set of all elements which belong to either or both of the pair of sets. The symbol for union is as follows:

$$A \cup B = C$$

If A and B are sets, then A union B equals set C. Set C is the set composed of all the elements in either A or B or both. For example,

if $\quad A = \{a, b, c\}$ and $B = \{d, e, f\}$

then, $\quad A \cup B = \{a, b, c, d, e, f\}$

Also, consider these sets:

Let $\quad X = \{1, 2, 3, 4\}$ and $Y = \{3, 4, 5, 6\}$

then $\quad X \cup Y = \{1, 2, 3, 4, 5, 6\}$

Union is sometimes called *sum. Make up other examples of sets and their unions.*

Intersection of sets. By *intersection* of a pair of sets, we mean the set of all elements that belong to both the sets. The symbol for the intersection of a pair of sets is $\cap$. For example, A intersection B is written

$$A \cap B$$

Consider the sets $A = \{a, b, c\}$ and $B = \{d, e, f\}$.

Then, $$A \cap B = \{\ \}$$

the null set, since there are no elements common to A and B. Now, consider the sets $X = \{1, 2, 3, 4\}$ and $Y = \{3, 4, 5, 6\}$. Then,

$$X \cap Y = \{3, 4\}$$

Intersection is sometimes called *product*, but it differs from our usual concept of product. *Make up other examples of sets, and find their intersections.*

Sets which have no elements in common are called *disjoint sets*. Disjoint sets have the null set as their intersection.

Complementary sets. The *complementary* set of a subset is composed of those elements that remain when the subset is removed. Consider set $A = \{a, b, c\}$. Then $B = \{b, c\}$ is a subset of A. The set $C = \{a\}$ that remains when set B is removed is called the set *complementary* to set B over set A. Sets B and C together make up set A. The term *complementary* is used in the same sense as complementary angles in which the pair together make up a whole.

The universal set. Quite frequently we desire to have a general set containing all the elements of other sets. This set may be defined in various ways and is called the *universal* set. For example, we may let the universal set be the letters of the alphabet. Then all sets under consideration would have only elements, which are letters of the alphabet. Examples are

$$\{a, b, c,\} \qquad \{a\} \qquad \{x, y\}$$

We denote the universal set by U. If we have a set A whose elements are taken from the universal set U, then the complement of A, written $\bar{A}$, consists of all those elements in U not in A.

Intuitive concepts of sets. A visual interpretation of sets may be made by drawings, usually called *Venn* diagrams. The points in a large rectangle

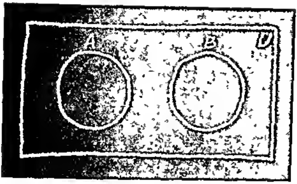


Figure 2.1

may be considered the universal set U. Sets may be represented by circles within the rectangle, and the elements of the sets may be thought of as the points within the circles. Such a representation will depend upon our intuitive concepts of points and sets of points, since a definition of points and related ideas is as difficult as a definition of number.

Several cases of the union and intersection of sets will be illustrated. Also, subsets and complementary sets will be demonstrated. Shaded areas indicate the union and intersection of sets as well as the complement of a set.

Case 1. Let the points in the rectangle (Fig. 2.1) be the universal set U. Let circles A and B represent sets within U which have no common points. Then, A union B (written $A \cup B$) is the set of points consisting of all points in set A and all points in set B. The shaded area (Fig. 2.2) represents $A \cup B$. Now, A intersection B (written $A \cap B$) is the set of points consisting of all points in both set A and set B, that is, the points common to A and B. Since A and B have no points in common, the intersection of A and B is the null set (Fig. 2.3).

Case 2. Let the circles A and B (Fig. 2.4) represent sets which have some points in common. Then $A \cup B$ consists of all points in A not in B, all points in B not in A, and all points common to A and B. The

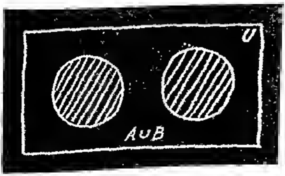


Figure 2.2

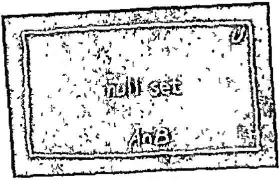


Figure 2.3

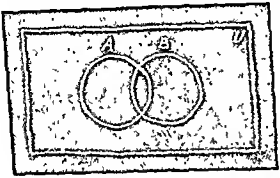


Figure 2.4

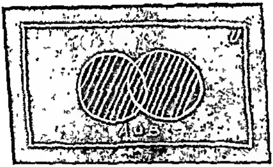


Figure 2.5

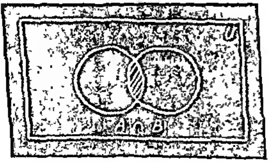


Figure 2.6

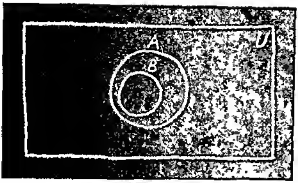


Figure 2.7

shaded area shown in Fig. 2.5 represents $A \cup B$. Now, $A \cap B$ consists of all points common to A and B. The shaded area (Fig. 2.6) represents $A \cap B$.

Case 3. Let circles A and B represent sets such that B is a proper subset of A. Then circle B (see Fig. 2.7) is entirely within circle A. The union of A and B, written $A \cup B$, is set A in this case (Fig. 2.8). The intersection, $A \cap B$ is set B (Fig. 2.9). The complement of set A, written $\tilde{A}$, is that part of the universal set outside of A (see Fig. 2.10).

Equivalent sets. Consider a pair of skates, a pair of arms, and a pair of pencils. Each of these pairs forms a well-defined set. What do these sets have in common? It is not their shape, size, color, or use. The only

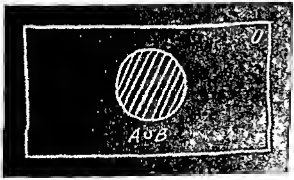


Figure 2.8

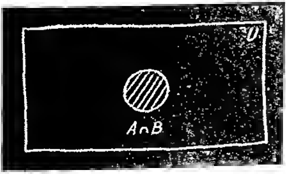


Figure 2.9

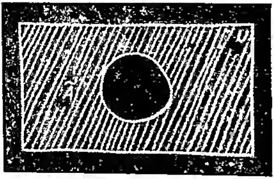


Figure 2.10

common characteristic is their plurality. The same thing is true of sets each of which contains a trio of objects.

How do we know whether a pair of sets have the same plurality? We do this by counting. If they have the same number, we say they have the same plurality or "manyness." Since we are now in the process of defining numbers, we cannot yet count or say they have the same number. Suppose we have a set of cups and a set of saucers. Place a cup in each saucer. If there is a cup in each saucer and no cups left over, then we know the sets have the same plurality. Notice that we have not said the sets were equal, only that they have the same plurality.

One-to-one correspondence. A pair of sets, A and B, are equivalent if to each element of A there corresponds a single element of B and if to each element of B there corresponds a single element of A. In other words, if we take an element from set A, then we may pair it with an element from set B. When the pairing is completed, there will be no elements left in either set A or set B. This is called *one-to-one correspondence.* For example, consider sets A and B:

$$A = \{a, b, c\}$$

$$B = \{x, y, z\}$$

One way of showing the correspondence would be

$$a \leftrightarrow x \qquad b \leftrightarrow y \qquad c \leftrightarrow z$$

with the double-headed arrow indicating a correspondence each way. Write all the ways in which a one-to-one correspondence may be set up between the elements of sets A and B.

Now consider an example of sets which do not have the one-to-one correspondence property.

$$C = \{1, 2, 3, 4\}$$

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$$B = \{x, y, z\}$$

Note that we may set up a correspondence between 1 and x, 2 and y, and 3 and z, but there is no element in B to correspond to 4. One element in C is left when we pair the elements of sets C and B. Therefore, we do not have a one-to-one correspondence between sets C and B. Sets C and B are not equivalent. One-to-one correspondence is a property possessed by equivalent sets.

There are two ways of writing equivalent sets:

$$A \leftrightarrow B \qquad \text{or} \qquad A \sim B$$

These are read "A is equivalent to B." Note that if $A \sim B$, then $B \sim A$; that $A \sim A$; and that if $A \sim B$ and $B \sim C$, then $A \sim C$. Any operation possessing these three qualities is called an *equivalence relation.* From its definition, *one-to-one correspondence* may be seen to possess these qualities.

FROM SETS TO NUMBERS

We are now ready to define number, or at least a certain kind of number. Notice that in all of our discussion of sets we have not used any number names. A preliminary definition of the number 1 was made. The numerals or symbols for numbers used in examples of sets could just as well have been any other kinds of marks on the paper. References have been made to "single" and "pair" with the expectation that these terms involving plurality would have meaning in the same way that other words in our language have meaning. Several other concepts and definitions are needed before we are ready to go from sets to numbers.

Successor of a set. Let us consider the set $A = \{a, b, c\}$. There are several subsets of set A. (See previous section on subsets.) You will recall that A is a subset of itself. Also, you may recall that sets may consist of sets. For example,

$$Z = \{\{a\}, \{a, b\}, \{a, b, c\}\}$$

Thus, the set $A = \{a, b, c\}$ may itself be an element of another set. Now, consider the set A as an element and construct a new set consisting of the elements of set A and set A, itself, as an element. This set would be

$$\{a, b, c, \{a, b, c\}\}$$

We shall call this set the *successor* of A, and designate it by the symbol "$s(A)$." Note that the set A is an element of the $s(A)$. *Write the successor*

of the set $X = \{a, b\}$. The *successor* of a set is the union of the set with a set whose only element is the set itself.[7]

Standard sets. Now, standard sets may be defined and names given to them. The first standard set is the null set $\{\ \}$.[8] Its name shall be *zero*, and it will be symbolized by writing

$$0 = \{\ \}$$

Now we may construct the successor of zero. This is the set composed of the elements of the null set (has no elements by definition) and the null set considered as an element. Therefore, $s(0) = \{0\}$. Note that we have a union of the following sets:

1. The null set $\{\ \}$
2. The set whose only element is the null set $\{\{\ \}\}$

That is,

$$s(0) = \{\ \} \cup \{\{\ \}\}$$

This may be written:

$$s(0) = 0 \cup \{\{\ \}\} = \{\{\ \}\} = \{0\}$$

The successor of 0 is a set consisting of a single element, namely, 0. This set we shall call *one*, and its symbol will be "1." That is,

$$1 = s(0) = \{0\}$$

The successor of 1 is the union of 1 and the set whose only element is 1. This may be written

$$s(1) = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\}$$

This set we shall call *two* and give the symbol "2."

The successor of 2 is the union of 2 and the set with 2 as its only element. That is,

$$\begin{aligned} s(2) &= 2 \cup \{2\} = \{0, 1\} \cup \{2\} \\ &= \{0, 1\} \cup \{\{0, 1\}\} = \{0, 1, \{0, 1\}\} = \{0, 1, 2\} \end{aligned}$$

This set we shall call *three* and give the symbol "3."

[7] Howard Levi, *Elements of Algebra* (New York: Chelsea Publishing Co. 1960), pp. 18–19.

[8] *Ibid.*

The next standard set is the successor of 3. It is the union of the set called *three* with the set whose only element is the set called three.

$$\begin{aligned} s(3) &= 3 \cup \{3\} = \{0, 1, 2\} \cup \{3\} \\ &= \{0, 1, 2\} \cup \{\{0, 1, 2\}\} = \{0, 1, 2, \{0, 1, 2\}\} \\ &= \{0, 1, 2, 3\} \end{aligned}$$

This set will be called *four*, and the symbol will be "4."

This process may be continued indefinitely, and names and symbols given to the standard sets. Note that the names and symbols given to the standard sets play no part in the definition of the sets. In other words, many different systems of numeration (written symbols) may be used to designate the standard sets.

Equivalence class. Having defined standard sets, we may now consider collections of sets. First, from our definition of standard sets, we may see that they are ordered. They are ordered in the order in which they were constructed.

$0 = \{\ \}$	zero
$1 = \{0\}$	one
$2 = \{0, 1\}$	two
$3 = \{0, 1, 2\}$	three
$4 = \{0, 1, 2, 3\}$	four

Now consider the set $A = \{a, b, c, d\}$. We can obviously put this set into a one-to-one correspondence with the standard set $4 = \{0, 1, 2, 3\}$. We could conceive of many sets which could be put into a one-to-one correspondence with 4. In fact, we could select many sets which could be put into a one-to-one correspondence with any other of the standard sets. The collection of sets which are equivalent to a standard set is called an *equivalence class*.

Cardinal number. An equivalence class is a collection of sets. The standard set used to define each equivalence class has been given a name and symbol. The *cardinal number* of a set is defined as the equivalence class to which the set belongs. The name and symbol for the cardinal number will be the same as the name and symbol assigned to the standard set used to define the equivalence class. Thus, the cardinal number of a trio will be the class of all trios. The class of all trios will be the cardinal number 3.

The cardinal numbers are ordered just as the standard sets from which they take their names are ordered. The terms "larger cardinal

number" and "smaller cardinal number" may be defined as follows: the set A has a larger cardinal number than the set B if the set B is equivalent to some proper subset of A. Zero is obviously the smallest cardinal number. The cardinal number 1 is larger than 0. Using the standard sets, $0 = \{\ \}$ and $1 = \{0\}$, as examples of sets having cardinal numbers 0 and 1, we see that the only proper subset of 1 is $\{\ \}$ which is equivalent to 0. In the same manner, we may show that

0 is less than 1

1 is less than 2

2 is less than 3

The symbol for "is less than" is "<," and for "is greater than" is ">." Therefore, we may write:

$$0 < 1 < 2 < 3 < \ldots < 99 < 100 < \ldots$$

Now the cardinal numbers are ordered and may be used to indicate the plurality of a set. If a set is empty, it is said to have the number zero. If the set is not empty, we assign the cardinal numbers in order beginning with 1 to members (elements) of the set. The last-named cardinal number is the cardinal number of the set. This is called *counting*. This answers the question of plurality or "manyness" of a set.

Is there a last cardinal number? From our definition of standard sets, equivalence classes, and cardinal numbers, we could go on constructing cardinal numbers forever. For any cardinal number named or defined, we could always construct a larger one. The set of cardinal numbers then is said to be an *infinite set*, or to contain an infinite number of elements.

Natural numbers. Some writers define natural numbers as abstract symbols whose meaning lies only in the rules used to manipulate them.[9] Others simply use "cardinal numbers," "natural numbers," and "counting numbers" to mean the same entities.[10] In fact, our cardinal numbers will simply be called *numbers* until other kinds of numbers are defined, and the kind must be specified in order to eliminate doubt as to the kind of number meant. Other terms used to refer to our cardinal numbers are "whole numbers" and "non-negative integers."

[9] Irving Adler, *The New Mathematics* (New York: The John Day Company, Inc., 1958), p. 35.

[10] E. Glenadine Gibb, Phillip S. Jones, and Charlotte W. Junge, "Number and Operation," *The Growth of Mathematical Ideas*, Twenty-Fourth Yearbook, National Council of Teachers of Mathematics. (Washington, D.C.: The Council, 1959), p. 14.

LOGIC AND LEARNING

In the elementary grades, children learn to count by memorizing the number names and by assigning the number names in order to each member of a set of objects. The first process is called *rote counting*; the second is called *rational counting*. Later, children learn to assign numbers to sets of objects by the way they are grouped without "counting" each object. Obviously, there is a limit to how far these concepts of number may be extended. Large groups cannot easily be assigned a number. In fact, we can conceive of groups which we could not count in a lifetime or in any finite period of time. The way by which children learn number concepts, by using objects and learning number names in many different contexts, is a perfectly good one. Only relatively elementary number concepts, however, can be associated with objects in the manner by which children learn. Abstractions and the use of symbols must be learned if the more advanced ideas in mathematics are to be a part of the child's knowledge.

How many people can give a satisfactory answer to the question, "Why do we invert and multiply in order to divide by a fraction?" or "Why does a minus times a minus give a plus?" Relating these concepts to physical objects is difficult. A logical development of numbers will help to provide an explanation of some of the operations performed on numbers. Also, such a development will help us distinguish between operations on numbers and the processing of numerals. Many of what are called *number operations* are simply processes peculiar to the kind of numerals being used. A better understanding of number and number operations can result from a close look at one of the logical ways of developing number concepts.

Later chapters will show how the number operations and other types of numbers, such as the integers, rationals, and real numbers, may be logically developed.

Something to Think About

1. Can you define number? Write a one-paragraph description of what you think number is.
2. Which term, *mathematics* or *number* is more inclusive? Explain.
3. Exactly what is the difference between a number and a numeral?
4. Explain how children may come to confuse numbers and numerals. Give several examples.
5. What is meant by "intuitive" concepts of number?

6. What are the necessary components of a number system?

7. List the basic ideas behind the Hindu-Arabic system of numeration.

8. Try performing some number operations using an additive system of numeration. A ciphered system.

9. How may positional systems of numeration with bases other than 10 be of use in understanding our base-10 system of numeration?

10. Make up examples of sets. Represent them symbolically. Find the union and intersection of these sets.

11. Illustrate some examples of sets and set operations by use of Venn diagrams.

12. What advantages does a logical development of number have over other ways of conceptualizing number?

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PART TWO / ARITHMETIC IN THE PRIMARY GRADES

Arithmetic in Grade O

CHAPTER THREE ARITHMETIC IN GRADE ON

The title of this chapter does not imply that there is a generally accepted scope of arithmetic activities assigned to first grade. Indeed, it is doubtful whether any two arithmetic programs would agree with each other on suggested content for any grade level. The authors, however, hope that, out of this diversity, they can present some fairly typical content topics at various grade levels, along with some information regarding methods of teaching applicable to each.

This chapter includes the following topics:

A. Readiness Tests

B. A First-grade Arithmetic Program

C. The Slow Learner

D. The Rapid Learner

A. Readiness Tests

As was mentioned earlier, many students enter first grade with very little background for, or interest in, number study. Such comparative terms as longer and shorter, larger and smaller, or more or less have been adequate for most purposes; parents or older siblings have taken care of the more quantitative situations. Some small children know when their favorite television show comes on, but usually cannot read time from a clock or watch. Number songs and games are sometimes learned from parents or phonograph records. These most commonly stress rote counting (one, two, buckle my shoe), and this is easily forgotten. Hence, the entering first-grade student usually has little background in quantitative work, mainly because he has not felt a need for skills in this area of learning.

Some school systems, public and private, are giving considerable attention to nursery-school and kindergarten training for preschool youngsters. Generally, it seems that number work does not get major emphasis in such training, since most sets of arithmetic texts do not include a "primer." Hence, we may safely assume that the number work in many preschool classes is "incidental and accidental."

On the other hand, it is not particularly unusual to find a few students in a class of beginners who, through parental teaching or because of attendance at an arithmetic-oriented kindergarten, are well started on number fundamentals. Such students might count rationally to 10 or above, recognize groups of objects in a variety of arrangements, or know the basic concepts of addition and subtraction.

From the foregoing, it can be seen that pupils at a particular grade level do not start from the same place, not even in the first grade. Hence, early in the first year, many teachers like to give some sort of readiness test, since this provides a point of departure for the work of instruction.

There are some standardized readiness tests available, although they do not seem to be used very extensively. The first-grade teacher has little interest in such features as national norms for such a test. Rather, her interest is in learning where she should start with number work for the students in her class. Hence, many teachers prefer to improvise a test suited to their own student groups.

Such a test is described by Hollister and Gunderson.[1] They recommend testing on the following aspects of number work: ability to count,

[1] George E. Hollister and Agnes G. Gunderson, *Teaching Arithmetic in Grades I and II* (Boston: D. C. Heath & Company, 1954), p. 24.

ability to recognize number quantities, ability to match number symbols, and ability to recognize the number symbols.

Such a test can be adapted to local needs and can be administered individually in ten minutes or less to most students. Of course, teachers would want to keep a simple record of the results of such a test.

What is the principal function of such a test? Essentially, it tells the teacher something of the background of each pupil, thereby permitting her to start each at a level suited to his needs and abilities.

B. A First-grade Arithmetic Program

Many people recall that, a few decades ago, the first task confronting beginners in arithmetic was to learn to count by rote to 100. Indeed, for some students, this *was* the first-grade program, since progress in this abstract assignment was sometimes very slow. One student illustrated how much such a chore meant to him when he mentioned that he was trying to learn the *alphabet* to a hundred. Now, it is generally accepted that assignments are carried out with greater efficiency if students are given tasks that they can perform in a reasonable length of time and if they see a reason for doing the work involved. The job of learning to count by rote to 100 failed to meet these criteria: it was a fairly long-term assignment, and students saw little use for it. Out of this has grown the realization that a fairly specific program of instruction is needed at first-grade level.

UNDERSTANDING THE NUMBER SYSTEM

There are as many different approaches to teaching an understanding of the number system as there are textbooks in arithmetic. Hence, no attempt is made here to point out a "best" way. Obviously, most teachers will be influenced by the text, work-text, or other such materials available to them.

One-to-one correspondence. One approach is to begin with a very simple type of quantification, that is, one-to-one correspondence. Here, a student is called upon to match objects in one group to objects in another group, without any particular effort being made to associate number names with either group. This might involve such activities as giving a pencil to each student or matching books with cards. In many programs, the student then moves to one-to-one comparison, still a rudimentary form of quantitative thinking. Here he determines

whether there are enough balls for each student to have one or whether, on a worksheet, there are as many lollipops as children. Later, of course, he will recognize this as a rudimentary form of subtraction. From here, he would logically move to rational counting.

Rote and rational counting. Perhaps it should be mentioned again that rote counting, the mere calling of number names, serves little purpose in a modern arithmetic program and hence is given a minimum of attention. From the first, however, children are given work in rational counting, that is, the counting of objects. This ability is basic to further progress.

Although it is still recognized that number work is essentially abstract in nature, we know enough of the learning processes of smaller children to realize that abstractions are difficult. Hence, in the earlier phases of number work, a great deal of attention is given to the use of concrete objects. Since rational counting involves counting things, most teachers make extensive use of the concrete in teaching this type of counting.

It is recognized that, although texts or teachers' manuals can suggest games or activities for students in which they work with concrete objects, essentially only the teacher is in position to handle this phase of instruction. The very best the text can do is to provide pictorial and other semiconcrete representations of objects.

In the early phase of work on rational counting, teachers use many forms of materials—bottle caps, books, popsickle sticks, bean bags, and the children themselves. Many games and activities can be built around this type of work. Later, teachers usually guide youngsters into a slightly less concrete phase, where they count steps, handclaps, or something along this line. These, of course, can be "experienced" but cannot be touched or handled as concrete objects.

As rapidly as the students can progress, teachers lead them into rational counting of semiconcrete or pictorial materials. Here, texts, workbooks, worksheets, cards, and many other types of materials are available to help the student. The "twoness" of a pair of shoes is shown by pictures rather than the actual shoes.

As is true of all teaching-learning processes, progress in the study of rational counting is an individual matter. Many programs try to develop facility in meaningful counting to 100 during first grade.

Cardinal and ordinal numbers. Since rational counting yields "how many" types of numbers, cardinal numbers result from such counting. Hence, cardinal numbers are taught along with rational counting. At

the concrete level, students are called upon to count out eight sheets of paper or twenty pencils. Semiconcretely, they are asked to tell how many monkeys are in the cage pictured in their book. Even less concrete would be the task of taking twelve steps forward. Most texts give many suggested activities for helping students with cardinal numbers.

Ordinal numbers describe a position in a sequence. Generally, we associate such terms as *first*, or *third*, or *tenth* with ordinals. There are exceptions, however. For example, when Johnny gives his birth date as March 14, 1955, he is using numbers in the ordinal sense.

It is usually accepted that ordinal usage is difficult for children. Many students who could count out twelve marbles would not be able to point out the seventh one counted. Consequently, most of the first-year programs do not go beyond "tenth" in ordinals. Some go only to "fifth."

Incidentally, one source of difficulty with ordinals is the names used. Although the relationship between "six" and "sixth" is fairly obvious, the relationship between "one" and "first" or "two" and "second" is somewhat obscure—hence, difficult to comprehend.

Of course, teaching materials that can be used for any other type of counting activity can be used in working with ordinals. Also, many teachers find natural teaching situations to use, such as the lunchroom line. The usual competition to be first or second in line can be helpful. One little boy was told by his teacher that, because of misbehavior, he would have to go to the last place in line. In a moment, he was back to his old position, explaining brightly that he couldn't be last since that place was already taken.

Number structure. The study of number structure as applied to the numbers below 10 appears to serve little purpose. True, there are certain legends regarding the origin of some of the number symbols ("2" evolving from "="), but the actual study of number structure starts with the two-digit numbers.

One of the most widely used techniques here is place-value pockets. Several types are available commercially, such as a sturdy plywood model sold by the John C. Winston Company. Many teachers, however, prefer to improvise them, using such simple materials as 8- × 11-inch envelopes. These can be compartmentalized with staples, with the compartments labeled "ones" and "tens." These are readily mounted on a tack-board for class demonstration (Fig. 3.1).

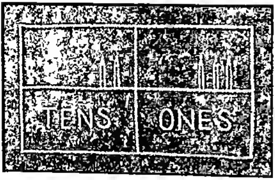

Figure 3.1

Some teachers get good results in number structure by having students outline a "ones" block and a "tens" block on their desks, using ordinary chalk. With an adequate supply of counters, students can follow at their desks the demonstrations being shown by the teacher.

Many teachers introduce this phase of work by adding counters—pencils, popsickle sticks, or other objects—in the "ones" pocket until they reach ten. Then, through questioning, they lead the class to understand that this pocket is overcrowded. Suggestions are invited as to ways out. Usually the class will provide the plan that some of the counters should go into the "tens" pocket. Some teachers make this a moment of drama, when they take 10 ones and, by a grouping device such as a rubber band, change them to 1 ten. Obviously, this, now that it is a ten, belongs in the "tens" pocket.

Activities of various sorts can be built around the place-value pockets. For example, a student might volunteer to put 16 in the pockets. Having placed a ten and 6 ones in the appropriate pockets, he might "prove" his work by holding the pockets against the board and writing a description of his number. Since the "tens" pocket contains 1 ten, he writes a 1 above this compartment. Since the "ones" pockets contains 6, he writes a 6 above it (Fig. 3.2). He has now described the contents

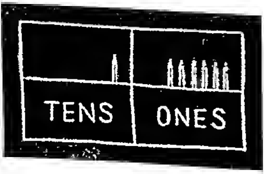

Figure 3.2

of the pockets by writing 16 while seeing them visually. Many similar activities can be used. Since it is the system that is being emphasized, most teachers concentrate on the tens, twenties, and thirties during first grade.

As is generally true in number work, students are led forward from the concrete toward the abstract as rapidly as they can move. For example, after working with sticks in place-value pockets—about as concrete as it can get—students are usually moved into semiconcrete work in the same area. This might be illustrated with the student who is asked to draw 16 circles on the board. Then he is led, through class discussion, to see *a ten* and 6 ones in the number. Sometimes, the first ten circles will be enclosed in a box or compartment in order to bring out more clearly that this is 1 ten. It is of vital importance that the students see the difference between *a ten* and 10 ones.

Also, at this stage children can work on the smaller numbers in a number chart. Later, this will evolve into the hundred board or hundred-chart. Since much of the early work starts with the concrete, there is no particular use for the zero as a point of departure. The first contact with zero usually comes in the writing of 10, and here it is treated as a part of a symbol, not as a separate mathematical symbol. Consequently, in building decades for a number chart, we normally begin the first decade with 1 and go through 10. Subsequent decades, of course, follow the same pattern.

It is unfortunate that the first two-digit numbers studied depart in a rather illogical manner from the usual nomenclature. The fact that "eleven" is believed to signify "one left" while "twelve" means "two left" would normally not be presented to students. Even the teens are somewhat confusing, especially because the "teen" comes last. That is, one would logically expect that, in "14," the "4" would be written first, since it is first in the number name. Since there is little likelihood of any changes occurring in such terms, we have no choice but to teach them as they are.

Of course, after explanations, laboratory work, and other such activities, a certain amount of drill (frequently called by a more palatable term) is necessary in order to help youngsters master the concept that in 26, the 2 describes the number of tens, and the 6 describes the number of ones composing the number.

Grouping and recognizing groups. Among the more difficult tasks in first-grade arithmetic is the teaching of group recognition. Yet, there can be no question as to the importance of this type of skill. Group

recognition is a part of several readiness tests, and many programs introduce it very early in first grade. Semiconcrete groups ("How many monkeys in the picture?") are studied, and teachers are encouraged to use concrete materials for the development of this concept.

Almost from the first, teachers observe two different approaches which students use in working with groups. Some students look at five objects in a group or in a picture of a group and, without conscious effort, see it as five. Others continue for months to operate at a much lower level by counting "one, two, three, four, five." Those who continue to see the individual members rather than the group will probably tend toward finger-counting when they reach the study of addition.

There is no general agreement as to best methods of teaching group recognition. Indeed, many students make the transition—from seeing ones to seeing groups—without knowing just how it happened. First-grade texts or work-texts, however, usually give many exercises in seeing groups in varying degrees of abstractness.

In some classrooms, dominoes or domino cards are much used in teaching group recognition. One precaution needs to be observed here. In dominoes, a five, for example, *always* has the same pattern. If this type of material is used repeatedly, there is danger that the student will come to recognize the pattern rather than the group. Hence, it is important that cards, charts, or other materials show groups in a variety of patterns (Fig. 3.3).

Concurrently with, or immediately after, the work on recognizing groups, students study how to make groups. This, of course, involves a variety of approaches. Some of these are: "Get enough pencils from the cabinet so that each child in your group will have one" (concrete); "Draw a circle around five goats in the picture" (semiconcrete); or "Draw a line under the group with four dots in it" (less concrete than pictures of animals). This type of activity leads logically and almost painlessly into work in addition and subtraction.

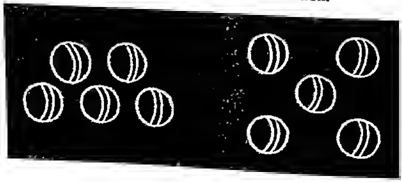

Figure 3.3

Incidentally, the importance of grouping and of the ability to see groups is illustrated in some of the programed materials available for self-instruction. Many such programs, designed for instruction in processes considerably more advanced than first grade, begin with having students recognize, and later draw groups. For example, in the program for self-administered work in multiplication and division facts (published by Teaching Machines, Inc., Albuquerque, New Mexico), students begin with group recognition, the implied assumption being that this skill is basic to the development of the higher skills.

Another aspect of group work is counting by groups. Certain programs require that students develop some facility in counting by twos, fives, and tens in first grade. These are usually introduced concretely, but go rather rapidly toward pictorial material. To many students, counting by twos is the most difficult of these three types of group counting. Such natural pairing situations as counting ears, eyes, feet, or hands in a group of students can add interest to this operation. Also, such grouping devices as egg cartons are frequently used. Counting by fives and tens frequently makes use of coins (nickels and dimes) as a familiar group-counting situation.

Associating number names, symbols, and groups. A crackerbarrel philosopher once described how "impossible" arithmetic was to him by telling about how he had to learn it in three languages. He said that his first chore was to learn the word "two" and to learn where it fitted into a sequence of such words. Then, he said, the teacher suddenly confronted him with the same thing in another language by writing the numeral 2 on the board and insisting that he use it. Then, having learned the word and the numeral, he was told that he actually didn't know what two was until he could count out two objects. This particular individual insisted that he "knew he was licked" when this happened.

Frequently, it is difficult for adults to see the complexity just pointed out, the complexity of teaching names, symbols, and rational counting simultaneously. In most modern programs, however, all these approaches are so closely coordinated that the student is not aware that he is doing several different things at once. Many arithmetic programs start, almost from the very first day, presenting pictures of groups, along with the descriptive number word and the number symbol. Many parallel aids, such as flash cards, charts, and assorted games, can be used to further this complex job of "learning in three languages."

Of course, the first phases of this work are concerned with recognizing number names and numerals; usually, however, and as an integral operation, students are introduced to the task of *writing* the numerals. This is frequently troublesome because (1) there is no way for a student to "figure out" why the symbol "4" stands for the same thing as the word "four," and (2) some degree of muscular coordination is required.

Several different approaches are available to aid in this task. For years, first-grade teachers have used commercially available charts for letters and numbers. These, in characters several inches high, are frequently mounted above the chalkboard, presumably to serve as models for the students. Also, of course, the teacher uses every opportunity to write the number symbols on the board, pointing out how each is correctly written (Fig. 3.4).

Certain first-grade teachers have developed or adapted verses, jingles, or songs to help students learn to write the number symbols. These are usually built around the separate motions used in writing the symbols. For example, one such jingle describes, with much repetition and many accompanying motions; "A line straight down and that is all, to make the number 1." Two is described as "half around and straight across"; three is "half around and half around." Four is "down, across, and then straight down." Five is "across, down, and half around." Six is "a line straight down, then all around." When used, of course, such jingles are rapidly superseded. It is not inconceivable that these could become a crutch, with a third- or fourth-grader still having to sing the jingle in order to write the symbol.

In order to involve the sense of touch in teaching students to write number symbols, some first-grade teachers make extensive use of cardboard or wood cutouts of the numerals. Such numerals, made of bakelite, are commercially available, but can be improvised by teachers. Usually, the numerals are large, possibly up to 6 inches high. It has been found that youngsters who are having trouble in writing number

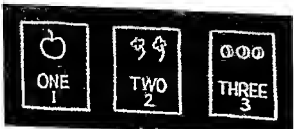

Figure 3.4

symbols frequently get a great deal of help by tracing such figures with the fingers. Variations of this technique would be giving the figures a light coat of plastic material from a pressure can and sprinkling lightly with sand, or gluing sandpaper to them. This adds more "feel" to the symbols and seems to increase their effectiveness as a teaching aid.

Tracing is a widely used approach in teaching children to write numerals. Many workbooks and work-texts use dotted symbols, usually with arrows showing the direction of motion for each part of each symbol. Many students will require supplemental practice on this type of work. This is usually provided by use of teacher-prepared worksheets or some other type of duplicated material. Further, as was mentioned earlier, teachers must accept the fact that skill in writing these symbols comes slowly and hence is developed over a period of years.

Some number symbols give much more trouble than others. For example, many students have trouble with the 5. Despite their best efforts, it comes out looking like a capital S. Probably even more troublesome is the 8. This symbol, of course, requires a downward stroke followed by an upward motion. Also, and frequently more of a problem, it has two separate and distinct segments. Hence youngsters tend to make it twice as large as the other numerals. Occasionally, a student will insist on reversing a symbol; 3 and 7 are common offenders in this respect. The correction of such difficulties usually requires additional practice under close supervision by the teacher. Sometimes it may be desirable to break down the difficult numeral into parts in order to help students develop the desired facility.

Aids available. In working with small children, teachers constantly try to make effective use of a wide variety of teaching materials. It should be pointed out, however, that many of the popular, commercially available materials can be improvised by teachers or even by the first-graders. Indeed, such materials might well prove to be more effective than the "store-bought" models. Also, teachers should remember that most materials are simply aids; even their strongest advocates do not claim for them any virtues not implied in the term *aid*.

Obviously, a complete list of the available aids cannot be included here, since it would be quite long, and new devices are constantly being developed. Hence, this discussion will deal primarily with some typical aids for first-grade arithmetic.

A booklet by Berger and Johnson[2] gives a good discussion of the effective use of arithmetic aids. It also includes lists of devices, models, and related aids, along with names and addresses of suppliers.

Some typical aids for use in helping first-graders to understand our number system are magnet boards with numerals, sold by Academic Aids, Princeton, New Jersey; counting frames, such as those sold by Milton Bradley Company, Springfield, Massachusetts; the two-place number board, such as that sold by Ideal School Supply Company, Chicago; and the primary spool number board, sold by the John C. Winston Company, Philadelphia.

A number of audiovisual aids are available for this phase of arithmetic. For example, there is a series of filmstrips on "Using and Understanding Numbers" for grade one by Stanley Bowmar Company, Valhalla, New York. The first three filmstrips in this series are useful for developing an understanding of our number system. Several films are also available, for example, Coronet Instructional Films, Chicago, sells a film entitled *Let's Count*, which is designed specifically for teaching such points as the difference between ordinal and cardinal numbers.

SEMIQUANTITATIVE COMPARISONS

Since quantitative thinking is foreign to most small children, many arithmetic programs give considerable attention to the semiquantitative concepts that logically introduce number manipulation. It would not be technically correct to refer to this as an area of study. On the other hand, such work could be thought of as contributing to readiness for quantitative thinking.

One first-grade text[3] builds activities around such terms as (1) more, as many as; (2) top and bottom; (3) left and right; (4) first, next, last; (5) larger and smaller; (6) more and fewer; (7) above or below; (8) longer and shorter; (9) more or less. As one would expect, these terms are used in a variety of activities, usually associated with pictures.

Another text[4] uses some of the comparison terms just listed. Some others used are (1) long, longer, longest; (2) short, shorter, shortest; (3) tall, taller, tallest; (4) high, higher, and highest.

[2] Emil J. Berger and Donovan A. Johnson, *A Guide to the Use and Procurement of Teaching Aids for Mathematics* (Washington D.C.; National Council of Teachers of Mathematics, 1959).

[3] The Silver Burdett Series, of which R. L. Morton is senior author.

[4] The Allyn and Bacon Series, of which C. Newton Stokes is senior author.

One might expect that the teaching of semiquantitative comparisons might be limited to the early part of the school year. Although usually more attention is given to it early in the year, such work continues throughout the school term. Comparisons are introduced wherever they seem to fit in best. Also, one must keep in mind that even the simplest concepts must be repeated at intervals. A teacher can seldom write something off the agenda permanently on the ground that all her students have mastered it.

ADDITION

As has been pointed out earlier, one of the most important goals in first-grade arithmetic is the development of an understanding of number. In many arithmetic programs, however, activities along this line are combined with the teaching of the fundamental operations. The premise, of course, is that there are mutual learnings in the study for understanding and the development of skills.

The adept teacher seldom precedes the introduction of addition with any formal notice. We can readily imagine the response of some pupils upon hearing, "Today we start the study of addition." Instead, the idea of "How many in all" is brought in so gradually that most pupils do not realize that something new is being studied.

A key concept in addition is that students be able to "see" groups. Since addition is essentially a matter of regrouping, most teachers devote considerable time and effort to the development of skill in working with, describing, and recognizing groups before they move to addition. If the child is to make satisfactory progress in addition, certain other concepts must have been mastered. He should be able to do rational counting, recognize number symbols, write number symbols, and reproduce ("count out") groups of designated sizes.

Concrete approach. The concept of addition, which, many years ago, was introduced as a definition, is usually presented concretely. The sequence of events frequently follows this pattern: (1) study of *a group*; (2) study of two separate groups; (3) study of a new group formed by combining the two separate groups. (See Fig. 3.5.)

Resourceful teachers can find many ways to introduce this operation. One of these is through dramatization. What, for example, could be more concrete than to have Mary and Ben (one group) walk across the room to join Sally, Bobby, and Joe (another group) to form a single group? Numerous activities of this type can be devised by teacher and

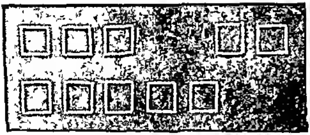

Figure 3.5

students. Many such activities require that the "story" be written on the board. The opportunities for the teacher to exercise ingenuity are abundant.

A next step commonly used is moving from people to concrete inanimate materials. Pencils, books, bottle caps, and other such objects can be used, and the drama of combining two groups into one is repeated many times. Incidentally, these materials are less concrete than are people in the class only in that the individual objects do not have names.

Throughout these exercises, there should be ample opportunity for the students to improvise activities for the class to use. It is not unusual for members of the class to suggest variations that the teacher had overlooked.

Semiconcrete. First-grade teachers accept the principle that addition is basically abstract. They also recognize that children master abstractions best when those are preceded by concrete instruction. Hence, at this point, they make the transition from the concrete to the semiconcrete.

Many children find pictures of people to be the easiest of the semiconcretes. Hence, workbooks or work-texts commonly go from people (concrete) to pictures of people (semiconcrete). Stick men are more abstract than pictures of men; hence, this is a frequent next step. Pictures of toys, animals, and (always of interest) food are widely used. Ultimately, squares, triangles, circles, and other such figures, generally considered to be less concrete than pictures of more common objects, are given attention.

Of course, in dealing with pictorial material, students must participate at a somewhat higher level than with the concrete in that now the combining of two groups does not literally occur. The student has to "see" two airplanes join two other airplanes without actually bringing them together.

The abstract. Although there is no general agreement as to how much abstract work in addition, that is, the addition facts, should be given first-graders, can you imagine holding students on semiconcrete work when they are ready for the abstract?

Efforts are made to have first-grade students do abstract addition in both vertical and horizontal form instead of simply "4 apples and 2 apples are 6 apples." A frequently used method is showing the pictorial version and the symbolic version together for a considerable period of time, then gradually eliminating the use of the pictorial. Having made the transition to the abstract, teachers use many devices, such as flash cards, workbook exercises, or teacher-designed worksheets, each involving much repetition, in order to help the student become reasonably confident on a limited number of abstract addition facts. This, of course, is a *recurring experience*, a more palatable term for *drill.*

For years, some arithmetic programs have used the fifteen easiest addition facts in the first grade. This excludes the zero facts and goes from

$$\begin{array}{r} 1 \\ +1 \\ \hline 2 \end{array} \qquad \text{through} \qquad \begin{array}{r} 5 \\ +1 \\ \hline 6 \end{array}$$

This meant that first-graders did not work with sums greater than 6, and they learned 15 facts. Note that

$$\begin{array}{r} 4 \\ +2 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{r} 2 \\ +4 \\ \hline \end{array}$$

make a single "combination," but provide two addition "facts." This means that the first fifteen facts are

$$\begin{array}{ccccccccccccccc} 1 & 2 & 3 & 1 & 2 & 1 & 3 & 1 & 4 & 2 & 3 & 2 & 4 & 1 & 5 \\ 1 & 2 & 3 & 2 & 1 & 3 & 1 & 4 & 1 & 3 & 2 & 4 & 2 & 5 & 1 \\ \hline 2 & 4 & 6 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 & 6 & 6 & 6 \end{array}$$

Some educators, however, think that this program is too easy; some texts are going as far as sums of 10 in first grade. This amounts to a total of 45 facts. There is no right or wrong in such a procedure, since each teacher is dedicated to the task of bringing each student to the highest degree of advancement of which he is capable.

It should be pointed out that the transition from concrete to abstract is not a kind of "graduation," where the concrete is dropped completely

when the student becomes able to handle the abstract. It is a back-and-forth operation in which students, on occasion, may be using two or all three levels of thinking on the same example. Just as a physician cannot provide a prescription that would be good for all the first-graders in a room, the teacher cannot provide a teaching method that would be universally applicable.

Aids available. The most valuable teaching aids in this phase of arithmetic are available free. They are the various concrete materials that can be used to form simple manipulative groups. Some such items have been mentioned earlier, such as pencils, people, books, bottle caps. Some teachers use index cards; others use popsickle sticks. The wooden tongue-depressors used by physicians work well. Of course, many school suppliers sell materials for this purpose. A resourceful teacher, however, would find it very simple to use materials within her room.

Some teachers find that flash cards help in teaching addition facts. These are available from practically all suppliers, but may also be carried by your local drug or variety store. Some parts of "Addo Arithmetic Game," sold by Creative Playthings, Inc., New York City, could be used at this level. Also, the counting frame, available from most of the commercial supply houses, is quite popular among first-grade teachers. Frequently this aid is sold in two sizes, a large one for the teacher to use for demonstration and smaller ones for students to use at their desks. Creative Playthings also sells a set of addition cubes designed to give meaning to the addition process.

There are a few films and filmstrips available, but in most cases, the teacher must select the part that would apply. This would be true, for example, of Coronet's film, *Addition Is Easy.* Some filmstrips are "Using and Understanding Addition" and "Addition Combinations," both available from Stanley Bowmar Company. Young America Films, Inc., has a filmstrip series on arithmetic. Certain frames from some of these films would be applicable.

An interesting gadget for self-administered drill is the Arithmequiz, sold by Edmund Scientific Company of Barrington, New Jersey. This device, nominal in cost, requires that, for a particular addition fact, the student insert an electric prod to indicate his choice of the sum. If he is correct, his "reward" is that a small light starts glowing. Although the charts available with this are extensive in scope, the teacher can select the part she wants to use at any grade level. Variations of this device may also be available in local variety stores.

SUBTRACTION

As usually presented in first grade, subtraction follows very closely after the teaching of addition. In many programs, the two are fused so skillfully that students do not realize that a new operation is being introduced. After the early phases of subtraction, however, in order to minimize confusion, it is essential that the two operations be kept separate until students have become familiar with them.

Whereas there is essentially only one addition situation, there are several basic subtraction situations. Grossnickle and Brueckner[5] list four of these: (1) finding the remainder, (2) finding the difference, (3) finding how much more or less one group is than another group, (4) finding a part of a group. Normally, however, in first grade, only the "How many are left?" type of problem is used.

It is important that teachers see addition and subtraction as opposite processes, both built around the key concept of *groups*. Whereas addition means combining two groups into one, subtraction basically involves converting a single group into two component parts. Indeed, in first grade, the term *subtraction* gets little attention. The terms *take away* and *less* seem far more descriptive. Some teachers object to this as "watering down" the subject matter. There can be no question however, that "5 take away 3" or "5 less 3" means more to a first-grader than does "subtract 3 from 5." Hence, the terms *take away* and *less* seem desirable in forming first concepts of subtraction.

Concrete. As was the case in addition, the teacher usually introduces subtraction with concrete objects. Of course, this must be done by the teacher, since the text or workbook cannot go beyond suggesting objects that might be used. In fact, students may take part in a simple drama to illustrate subtraction. "If Mary, Jim, and Sarah are in a group at the chalkboard but Jim returns to his seat, how many are left at the board?" Obviously, the teacher and the class can improvise many variations of this.

The illustration of "take away" just presented can be used to show one difficulty in subtraction. Although Jim has changed his location, he is still very much in evidence. The key point, of course, is that, although Jim has not disappeared into thin air, he is no longer associated with his original group. (*Groups* again!)

[5] Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meaning in Arithmetic* (New York: Holt, Rinehart & Winston, Inc., 1959), p. 156.

MEASURES

One feature of first-grade arithmetic that has been mentioned repeatedly is that we seldom use numbers in the abstract. They are normally used in association with objects. Thus, a child who can readily visualize "six pencils" would have great difficulty trying to visualize just "six." (Can you visualize it?) Consequently, teachers have a "built-in" opportunity to work in certain simple types of measures, using them as concrete objects.

Money. A type of measure that is widely used by first-graders is that of money. Most of the students have had enough experience with small-denomination coins so that no air of mystery is involved. Many teachers limit their work to pennies (technically "cents"), nickels, and dimes.

The first type of work with money usually consists of recognition of coins, which is a "readiness" factor for further work. Although most students will recognize the three coins mentioned earlier, the teacher cannot safely assume this. If there is difficulty, some simple activity in handling coins will usually take care of it. Unfortunately, play money is of little value here; the real thing is necessary. The nickel usually is easy to recognize, since it is larger than the others. Then, of course, the penny and the dime differ in color.

After having established coin recognition, teachers usually try to develop an understanding of the fact that the coins are of different value. Some understanding is achieved incidentally through play activities. A "play store" is commonly used. Incidentally, several supply houses sell boxes, packages, and other items to be used in a play store, but the teacher and students can improvise these quite effectively. And, of course, if it should develop that in certain cases the incidental approach is not getting the job done, the teacher may use more formal activities in order to establish value relationships among the coins. After students have become reasonably sure of themselves as to the value of coins, play money can be used as the basis for a wide variety of teaching activities, including simple addition and subtraction, making change, and others.

Time. Perhaps one reason why we start teaching time at first-grade level is that the concepts involved are quite complex and must evolve over a period of years. Also, in a civilization such as ours, even small children "live by the clock" to a considerable degree. Time is difficult to teach in first grade for several reasons. First, there is no way to use

the concrete to any degree. After all, time is so abstract that it practically defies definition. (What is *your* definition of time?) A youngster can stack five pennies and see them as equivalent to a nickel, for example, but he *cannot* stack five minutes. A second difficulty in teaching time is that, to most beginners in school, the whole thing is of little interest. Did you ever watch a frantic parent trying to get an indifferent first-grader to school on time? The youngster, dreamily using twenty minutes to put on one sock, is completely unimpressed with such time-related pleadings as "We have to leave in five minutes;" "It's already eight-thirty;" or even "Hurry," or "You'll be late." To the child, time is a tiresome bother that seems to disturb adults to an unseemly degree.

Most first-grade teachers introduce time by using the classroom clock as a teaching material. Some teachers like to have a variety of clocks on hand during this phase of work so that the youngster won't associate time with one particular clock. Having studied the external features (dial and two hands) of the clock, the class might make a list of clock readings that mean something to them, such as the time of a favorite television or radio show, the time the school day starts, the time the class goes to lunch, and many others. Then the students, under the guidance of the teacher, show where the clock hands would be for each such time.

It is relatively easy to improvise clock dials for student use. One of the most popular involves writing the dial digits on the bottom of a paper plate. Long and short hands, cut from cardboard, are attached with a brad so that they can move. Children, working singly or in groups with such a device, can follow the work of the teacher or can improvise problems of their own. Although there is no standard pattern, many first-grade teachers do not try to teach time units smaller than an hour.

Many types of work-sheet activities can be provided students in their study of time. A few examples follow: (1) show the dial, with one or both hands missing and have the students draw hands so as to show a particular time; (2) have numerals or hands missing on a sketch, the student's job being to "find the missing part;" (3) have clock dials without hands and then have hands sketched in to show certain "key" events of the day, such as "time to get up," "time to go to school," and ultimately "time to go to bed."

Distance. The concept of distance is relatively concrete, especially when compared to time. In the first-grade program, it is common

practice to limit the study to a single distance unit, the inch. It would be preferable to have rulers without any subdivisions smaller than an inch for this work. Since such rulers are not commonly available, it is generally necessary to buy them from educational supply companies.

It would not be realistic to expect that first-grade students will develop much facility in measuring distances. One can, however, develop a few concepts, such as (1) an inch on one ruler is the same length as an inch on another; (2) correct measures require care and practice in the process of measuring; (3) when Bill and Joe measure the same object, they should get similar results. In short, the first-grade teacher frequently uses distance measures as a means of teaching the meaning of measurement. Of course, there are many laboratory materials in the typical classroom for use in distance measurement—books, blocks, desk surfaces, lines on the board or on paper, and a host of others.

Actually, many first-grade arithmetic programs give little emphasis to length measures. Some dismiss the topic with one or two pages of text space. Few programs go into units of measure other than the inch. As was mentioned earlier, however, the inch is a good unit to use in establishing some of the major principles of measurement.

C. The Slow Learner

A student in a teacher-training program asked the instructor, "Does this individual-difference business become a problem as early as first grade?" His answer was, "It is a problem, especially in first grade." Whatever the admissions policy in your state, there will be some students who were barely old enough to enter and others who *almost* made it last year; that is, there will be a spread of a year in chronological ages. A variation of one year in six could cause a considerable difference in maturity. Further, some students will come from homes that have developed self-reliance; others have developed parent-reliance. In short, every teacher in school has to cope with a great variety of individual differences.

It would be beyond the scope of this book to try to list slow-learner techniques that would be useful in teaching each topic of first-grade arithmetic. Nevertheless, a few general suggestions may be made. One, in working with the slow learner, the teacher should stick rather closely to a minimal program. If your text considers that the student

should learn the addition facts through sums of 6 as a basic requirement, you would probably conform to that standard. The slow learner should be encouraged to work intensely on the minimal program, with a great deal of drill work associated with it. It is not unusual to find lower-ability students who are especially anxious to "spread their wings" and jump into work that is far too difficult for them. Here, the task of the teacher is to direct the learning situation so that the slower students will *want* to do that which they need most to do, that is concentrate on a minimal program. The task is far from easy! The teacher, however, can get a great deal of valuable help from the teachers' edition of her text. Many such books will have suggestions "for those who need help" at the end of each major topic.

Another principle in work with the slow learner has to do with the progression from concrete to abstract. As has been pointed out earlier, our task is to move students toward the abstract as rapidly as they can make the transition. With the slow learner, this change is made very gradually. Problems in human relations arise when children who are still working with sticks or blocks observe their neighbors working away with numerals. Nevertheless, the platitude is true: learning is essentially an individual process. And until Jimmy is ready to put aside counters, it is a basic error to have him do so. Certainly, he should be encouraged to move toward the abstract, but not to the degree that he will feel that he is being pressured.

Another problem can arise in this connection, however. Jimmy finds things to be peaceful and serene as long as he is in the reassuring presence of his counters. Hence, he may be inclined to accept the concrete as a way of life. The observant teacher can detect this situation with little difficulty. There is a hazy, ill-defined distinction between a "teaching material" and a "crutch." In the situation just mentioned, the concrete materials actually impede learning and, hence, should be abandoned. Incidentally, a college student occasionally asks, "If sticks can be effectively used as counters, what is so horrible about finger-counting?" The answer, of course, is the point mentioned earlier. When the appropriate time comes, the sticks can be removed, but not the fingers.

In working with the slow learner, the teacher needs to keep in mind that success is vital to learning. Nothing is more discouraging to a student than to go along for days and weeks encountering nothing but failure, and this is as true of first grade as it is anywhere else. So the effective teacher sees to it that her work assignments are such that

each student experiences some degree of success. Such a practice does not mean that each student encounters *nothing but* success. This would be impossible in a practical learning situation.

D. The Rapid Learner

When, in 1957, Russia launched the first man-made satellite, a period of hysteria followed. Somehow, America's schools were blamed for our failure to "be first in space." Out of the wild charges and countercharges of this period, at least one wholesome change occurred: the rapid learner, for too long the "forgotten man" in our schools, began to get major attention. This has had an effect, even in first-grade arithmetic.

What are some characteristics of rapid learners and their learning? For one, the rapid learner should be expected to achieve at a level considerably above a minimal program. Further, the rapid learner can move from the concrete to the abstract quickly. Certainly, he should be expected to achieve more in less time than the average or slow learner. There is little reason to fear that he will get an inferiority complex because of lack of success. Indeed, wary teachers fear that the opposite might happen!

Considerable attention has been given to the development of horizontal enrichment activities for the rapid learner. *Horizontal* in this sense means that he learns more about first-grade arithmetic rather than going into work of a higher grade level.

Again, one source of help in working with the rapid learner is the teachers' manual that accompanies the text or the teachers' edition of the text. (If you don't have these materials, you will want to get them. They are extremely helpful.) Many of these aids have sections titled "Enrichment Activities," "Supplementary Activities," or the like, usually at the end of each topic.

Some effective types of rapid-learner work are developing scrapbooks or charts, developing original number games, designing teaching aids for class use, building simple cross number puzzles, and preparing reports on topics of interest to the class. The resourceful teacher can extend the list almost indefinitely.

We have said that it is vitally important that the slow learner be rewarded with some degree of success. It is equally vital that the rapid learner be challenged with learning situations suited to his ability. Just as the teacher must be alert to the slow learner who wants

to spend his time on impossible tasks, so must she be alert to the rapid learner who would be content with mediocre achievement.

Something to Think About

1. Suppose that the mother of one of your first-grade students asked why you gave a readiness test. How would you explain it to her?
2. Occasionally one hears the criticism that first-graders do not learn as much arithmetic as they should because too much of their time is spent in play activities. Do you feel that this is a valid criticism? Explain.
3. Have any of the recent periodicals carried reports of research dealing with first-grade arithmetic? Why not tell the class about it?
4. How would you, the first-grade teacher, explain why you haven't taught Johnny to "count to a hundred"?
5. Sometimes the complaint is made that too much attention is given the slow learner in primary arithmetic. How do you feel about it? Cite recent articles to support your position.
6. The same type of complaint is sometimes made regarding the amount of attention given the rapid learner. How would you respond to this?
7. How would you deal with a first-grade student who tells you that he is sure he will have trouble with arithmetic because both of his parents had trouble with it?

Selected References

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This journal is probably the most valuable single reference available to arithmetic teachers at all grade levels.

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One of the classic references in this subject area.

Spitzer, Herbert F. *The Teaching of Arithmetic*, 3d ed. Boston: Houghton Mifflin Company, 1961.

An influential book in this field.

Stokes, C. Newton. *Teaching the Meaning of Arithmetic.* New York: Appleton-Century-Crofts, Inc., 1951.

Chapter II deals specifically with a program for six-year-old students.

The teachers' edition or teachers' manual of the text in use may be a valuable reference.

Arithmetic in Grade Two

Content and methods of presentation of second-grade arithmetic vary somewhat among the textbooks. Also, as has been pointed out, the text content may vary considerably from that which the teacher presents. When we add the further complication that a class is likely to include slow, average, and rapid learners, each being taught at an appropriate level, we see that the term *a typical program* is meaningless. Consequently, reference to "a program for grade two," in this chapter means simply "a program"—no more and no less.

On the other hand, the second-grade arithmetic program described here is based upon programs that are in actual usage. The levels of achievement sought are real, not imaginary. Consequently, the "Arithmetic in Grade Two" described in this chapter has been compiled from what we considered to be the best of rather widespread practices among second-grade teachers.

Topics presented in this chapter are

A. Readiness and Diagnostic Testing

B. A Second-grade Arithmetic Program

C. The Slow Learner

D. The Rapid Learner

A. Readiness and Diagnostic Testing

A beginning teacher might well wonder how long this "readiness business" continues to get attention. It is common to make readiness something mysterious or alien to everyday life. But isn't teaching anything at any level based upon something else? And if this "something else," be it attitude, ability, or any of a number of other factors, be lacking, couldn't that student be described as "lacking in readiness?"

The tendency, then, to think of readiness as a problem primarily at first grade is fallacious. A ninth-grader whose arithmetic background is too weak to permit him to take algebra certainly isn't "ready"; or a college freshman whose high school mathematics background makes it impossible for him to do acceptable work in his freshman year lacks "readiness." Indeed, teachers of adults frequently encounter problems that fall under the general heading of readiness. Consequently, any teaching is or should be influenced by the students' degree of readiness for the new instruction.

READINESS TESTING

Several authors of texts in arithmetic give considerable attention to readiness testing early in the second-grade year. Others interpret the term as applicable to each major process rather than to a particular grade level. That is, in some programs, attention is given to readiness for multiplication or readiness for division.

Some types of readiness-testing activities frequently used early in second grade are (1) marking a particular number that is written in a series of numbers (this tests number recognition); (2) marking the largest or smallest number in a group (knowledge of number size); (3) writing dictated numbers; (4) completing a sequence of numbers; (5) marking diagrams to illustrate simple addition or subtraction stories; (6) carrying out addition and subtraction operations with little or no illustrative material. In short, these tests are specifically designed to help the teacher locate weaknesses that should be corrected before the students are introduced to new material.

It is doubtful whether you would ever find a commercial readiness test that would work for your class in your classroom as well as one that you could devise. If a second-grade teacher will keep in mind just what it is she hopes to do with such a test, her own test is likely to be effective. At this point, she has little need for national or regional

norms. Her chief interest is determining the degree to which each student is ready for new materials.

In an earlier chapter, it was mentioned that a readiness test in first grade would usually be an oral test administered individually. These limitations would not necessarily apply here. It should, however, be pointed out very clearly to the second-grade students that there is no occasion for grief or panic if they find parts of the test that give them trouble. (Yes, such things can be problems, even to second-graders.)

Further, there is no particular merit in giving a readiness test just because a teachers' manual recommends it. There is no virtue in giving such a test in order to file a copy of it in Mary's cumulative record. There is merit in such a test if Mary's work on it tells the teacher what Mary's strengths and weaknesses are *and* if the teacher's knowledge affects the type of arithmetic work that Mary is called upon to do.

DIAGNOSTIC PROCEDURES

Almost from the very beginning of arithmetic, there is need for diagnostic procedures. Readiness tests generally precede a new phase of work, whereas diagnostic tests usually come during or near the end of a phase.

Many textbooks include diagnostic tests at regular intervals. Many of these tests are designed so that a student who has difficulty with a particular example is referred to an earlier section for additional work of the type needed. Some texts begin such work as early as second grade.

In diagnosis, however, an observant attitude on the part of the teacher is even more valuable than tests. Since second-graders are forming work habits that will be major helps or handicaps, the teacher should strive to get good habits established. Some typical problems are the sprawler, the mumbler, and the finger-counter. The sprawler usually lacks self-confidence to the degree that he doesn't want the teacher to see what he is doing. So, while pretending to assume a casual posture, he actually succeeds in covering up his work. The mumbler finds it necessary to stall a bit or to talk things through in order to get properly oriented. So, in arithmetic he engages in a running soliloquy while hoping the correct procedure or result will come to mind. The finger-counter doesn't have quite enough confidence in himself to write 7 as the sum of 4 and 3 until he has given it a quick

check with his "built-in abacus." Some students become quite adept at finger-counting. Generally, all these practices associate themselves with lack of confidence and will usually recede as confidence is built up.

It is to be hoped that the second-grade teacher will be alert and observant as she watches her students work. Also, it would be well if, in checking the papers of her students, she would look for patterns of errors that tend to recur. Frequently, this can indicate points that need reteaching.

HOW MUCH RETEACHING?

The readiness tests and diagnostic procedures mentioned earlier will not be of much value to second-graders unless the teacher follows them with the type of teaching that is needed. And frequently, even at second-grade level, this means reteaching.

As our knowledge about how people learn has increased, it has become apparent that most of us do not learn much upon a first exposure. Hence, most arithmetic texts now devote considerable time at each grade level, after grade one, to reteaching material that was covered the previous year. This reteaching is done rather rapidly, but it is done. This has given rise to criticism from some sources to the effect that "it is all repetition." This practice, however, is based upon the knowledge that some repetition is vital to effective learning.

As applied specifically to second-graders, what does reteaching mean? If a particular student mastered the 15 easiest addition facts and the corresponding subtraction facts in first grade, could you safely assume that, after a summer vacation, he still knows these facts? And could you safely proceed to introduce new facts on this basis? Obviously, it would be unsafe to assume any such thing. The normal approach would be to reteach the 15 easy facts rapidly enough to keep the class interested, then proceed to new material. Also, there's always a good possibility that some of the students didn't learn a particular fact upon earlier contact. These students are learning it for the first time.

B. A Second-grade Arithmetic Program

It is more true of arithmetic than of any of the other common areas of study that work is cumulative. Each operation is developed spirally from previous learnings. This means that the second-grade program

includes (1) reteaching earlier work, (2) further development of first-grade topics, (3) the introduction of new phases of arithmetic.

UNDERSTANDING THE NUMBER SYSTEM

With the shift in emphasis in arithmetic from manipulation to understanding, more and more attention is being given to the development of an understanding of our system of numbers. Indeed, some topics in this area are included in each grade level, first through eighth.

Reteaching. Since many of the topics regarding number have been introduced in first grade, second-graders begin each phase of instruction with a fairly rapid restudy of these topics. This does not necessarily mean that there is a "reteaching" section in a second-grade text. Rather, a student who learned to count to 50 in first grade would still start with 1 in second grade—not 51.

Place-value concepts and zero. There is no general agreement as to when place value should be introduced to students. A few textbooks give only incidental treatment to this concept during grades one and two, with major emphasis in grade three. Since an understanding of place value is absolutely vital if two-place or three-place numbers are to mean anything, however, there is a trend toward a study of place value very early in arithmetic. Hence, the authors of this text discussed place-value pockets as a teaching aid in first grade. There is need for continued work in this area in second grade.

You will recall that in introducing the place-value concept, we built up 10 counters in the ones pocket, then objected to the fact that this position was overcrowded. The key moment came when, with a rubber band, we "converted" the 10 ones into 1 ten, this group now being placed in the tens pocket. There has been some disagreement as to the best way to symbolize this operation. Certain authorities have stated that a different type of counter, possibly of a different color, should be placed in the tens pocket, the idea being to symbolize the "oneness" of the ten. In the introductory phases, however, it is important that the students see the equivalence between 10 ones and 1 ten. For this purpose, grouping ones with a rubber band would probably be more effective than using a different type of counter.

Many second-grade teachers like to begin the study of place value with the pockets described earlier. Moving away from the concrete, however, they go to some variation of the place-value frame. This is

essentially a chalkboard sketch of pockets. Students use tally marks instead of counters, however, for this is slightly less concrete in nature. Teachers can build many types of activities around this frame. For example, if there are 15 girls and 14 boys in the second grade, the numbers can be shown on the frame as 1 ten, 5 ones, and 1 ten, 4 ones. Similar work can be used in building various number stories. The emphasis throughout, of course, is on the "ten-ness" of the 1 and the "one-ness" of the 5 and 4.

In earlier years, the teaching of zero was given attention from the very first in arithmetic. This, however, has changed considerably. Since zero as such defies concrete representation, and since small children have use for it only as it appears in 10, 20, or 30, second-grade teachers now give little emphasis to zero. When it first appears (in the number 10), most teachers present it as simply a part of the symbol, without going into its meaning as a separate entity. Of course, the teacher must ultimately point out that the zero in 10 means "no ones." Second-graders, however, would stand to profit little from an abstract discussion of this abstract symbol. Even teachers do not always agree as to the true nature of zero.

Another valuable aid in teaching the number system is the hundred-chart. This consists of the numerals from 1 to 100 arranged in decades on a large chart. Usually the numerals 1–10 will be in the top row. The numerals 11–20 are arranged below this group, and the order repeats for each decade. Some activities which have been used by teachers in working with the hundred board are (1) looking for systems or recurring patterns in the chart (a column consisting of 4, 14, 24, 34, 44, . . . , 94, for example); (2) figuring out the value of individual numbers, rows, or columns of numbers that have been covered; (3) analyzing various numbers as to how many tens and how many ones compose them; (4) looking for systems that would recur if the chart were extended to three-digit numbers. Also, a resourceful teacher would find this type of chart very helpful in answering students' questions or in assisting the student as he answers his own questions.

Larger numbers. In some programs, first-graders do not work with numbers above 100; others go somewhat higher. Serious study of three-place numbers, however, frequently comes in second grade. Of course, there are no new principles involved in such numbers. Some teachers like to introduce three-place numbers by using place-value pockets (the commercial models usually have a hundreds pocket).

Since this becomes a very cumbersome and time-consuming process, many teachers prefer to use the place-value frame instead.

There is no general agreement on how far a teacher should go in teaching large numbers to second-graders. Some texts go to 1000, but more commonly they go to about 200. It is argued that, once a student is clear on place value, he can understand the structure of numbers up to 1000, but he has very little occasion to deal meaningfully with such numbers. Hence, they would be of limited value to him.

Further work is usually given second-graders in group counting. Counting by tens and fives to 100 is commonly taught. Also, counting by twos (more difficult than the others) is given some attention. This is frequently done with concrete objects, and most programs do not go beyond 20 or 30. Incidentally, as a part of group counting, it should be stressed, at least in the introductory phases, that this can be a rational as well as a rote operation. Stacks of books, with 10, 5, and 2 in each stack, are well suited for such counting.

Although it would be hard to overemphasize the importance of having students understand the larger numbers as they study them, the more mechanical aspects also require attention. Such work as reading and writing these numbers should be included in the second-grade arithmetic program.

Ordinal numbers. As was mentioned earlier, ordinal numbers, which describe a position in an order or sequence, are generally more difficult than are cardinal numbers. The usual second-grade program includes the reteaching of the ordinals covered in first grade before expanding the list to include larger numbers. Some programs, however, do not recommend going to ordinals greater than "tenth" in second grade. Others go as high as thirty-first, on the assumption that the students have heard all of the dates in a month used in the ordinal sense.

In teaching ordinals, many teachers like to use position in a line, a date on a calendar, rows or columns of seats, and other familiar classroom situations. Generally, a good criterion to use in selecting activities for use in presenting ordinals is, "Does the activity help make ordinals more meaningful to the child?"

ADDITION

If we assume that the 15 easiest addition facts are presented in first grade, then one of the first jobs for the second-grade teacher is to reteach them. This usually is done quite early in the school year, and

many teachers prefer to begin with semiconcrete rather than with concrete aids. Should the teacher have a few students who are not ready for pictorial material, however, she would presumably encourage them to use counters for a transitional period. Many second-grade students should be able to use the addition facts with little attention to the manipulative materials.

How many addition facts? There has been some confusion as to how many addition facts there are. Most of the programs follow one of two popular patterns:

1. Since the zero facts, such as

$$\begin{array}{r} 4 \\ +0 \\ \hline 4 \end{array}$$

have little or no functional value for children, and since the zero can only be abstract, many texts deal with 81 facts. This is based upon the principle that, excluding zero, there are 45 possible combinations $\left(\begin{array}{r} 2 \\ +3 \\ \hline \end{array} \text{ and } \begin{array}{r} 3 \\ +2 \\ \hline \end{array} \text{ are a single combination}\right)$, ranging from $\begin{array}{r} 1 \\ +1 \\ \hline \end{array}$ to $\begin{array}{r} 9 \\ +9 \\ \hline \end{array}$.

Each combination provides two facts $\left(\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array} \text{ and } \begin{array}{r} 3 \\ +2 \\ \hline 5 \end{array} \text{ are different facts}\right)$, except that nine combinations are irreversible, these being the doubles $\left(\begin{array}{r} 2 \\ +2 \\ \hline \end{array}, \begin{array}{r} 3 \\ +3 \\ \hline \end{array}, \text{ and so on}\right)$. Consequently, from the 45 combinations, we get 90 facts. Then we subtract 9 for the doubles, giving 81 addition facts, excluding zero.

2. Another point of view is that we are dealing with a contrived situation in seeing

$$\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array} \quad \text{and} \quad \begin{array}{r} 3 \\ +2 \\ \hline 5 \end{array}$$

as separate facts, since the same basic thinking is involved in the two cases. This approach, of course, would yield 45 facts, excluding zero. This latter point of view fails to account for one basic phenomenon: it is quite common to find a student to whom

$$\begin{array}{r} 5 \\ +2 \\ \hline 7 \end{array}$$

is easy but who has difficulty with

$$\begin{array}{r} 2 \\ +5 \\ \hline 7 \end{array}$$

If the combination of 5 and 2 is identified as a single fact, would you say that the student has trouble with half a fact? In most texts, the 81 facts are presented as described earlier.

There is no general agreement about how many facts should be presented to second-graders. One pattern has been to present addition facts to sums of 6 or less in first grade. This list is retaught in the second grade, then expanded to include facts with sums of 10 or less. This represents 30 new facts for second grade, the assumption being that students will have mastered a total of 45 facts by the end of the second year in school. Some arithmetic texts, however, have expanded this list somewhat. It is not unusual now to find facts up to and including sums of 12 in the second grade. This would mean a total of 60 addition facts, including those taught in first grade. In a few programs, all the addition facts through $\begin{array}{r} 9 \\ +9 \\ \hline \end{array}$ are completed in second grade. Some teachers feel that this is too ambitious an undertaking for small children.

There is a commonly used pattern in teaching addition facts in second grade. This involves use of pictorial material, such as animals or people, to illustrate previously learned facts. Then exercises using less concrete pictorial material, such as blocks, stars, or triangles, are used to illustrate the same facts. Finally, only the number symbols, such as $\begin{array}{r} 3 \\ +2 \\ \hline \end{array}$, are used. Frequently, simple verbal problems accompany this material (Fig. 4.1).

When the time comes to introduce new addition facts, one common approach is to bring them in by families, frequently concretely. For example, a teacher might give each student 7 counters and ask him to see how many group patterns he can produce. After these have been set up, a student or the teacher could write the "story" of each pattern on the board, such as "John has $\begin{array}{r} 5 \\ +2 \\ \hline \end{array}$" and "Mary has $\begin{array}{r} 4 \\ +3 \\ \hline \end{array}$." Probably most of the facts yielding sums of 7 would be obtained. Texts and teachers' manuals offer many other suggestions as to methods of presentation. Teachers and students (Why not?) could devise others.

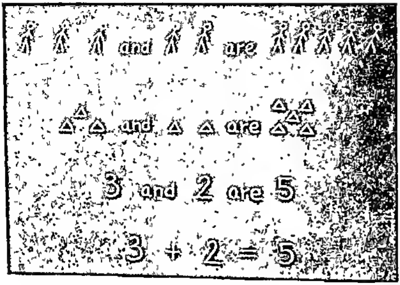

Figure 4.1

It will be noted that in many texts or work-texts, there is a great deal of repetition of facts. This, of course, is disguised drill, designed to help a student achieve actual mastery of the basic facts.

Column addition. It is relatively common practice to introduce simple column addition in second grade. This, of course, represents a major step forward and frequently causes some difficulty. This operation usually accompanies a number story and applies the same basic principle that the students used in adding two groups, that is, regrouping.

One of the chief problems in column addition is that the student must add a seen number to an unseen number. If we take an exercise such as

$$\begin{array}{r} 2 \\ 1 \\ +4 \\ \hline \end{array}$$

the student would add 2 to 1 to get 3. Then, without writing this 3 at all, he would add it to the 4 in the exercise. Many students find this very difficult to do (students of all ages, incidentally). Some are inclined to scribble the 3 (from $2 + 1$) in the margin and then combine it with the 4. This practice is to be discouraged, since it effectively changes column addition to 2-number addition. Also, when column addition is introduced, the teacher should be on the alert for finger-counting. Some students who have broken away from this practice may well revert to it when column addition is introduced. Another point to guard against is the tendency on the part of some students to

talk to themselves during the work. Instead of thinking "2, 3, 7," as we might hope, some will mutter "2 plus 1 is 3, 3 plus 4 is 7." Possibly this wouldn't be too objectionable in the very early phases of column addition; if permitted to continue, however, the student might well come to use this as a crutch. One means of detecting this roundabout way of thinking is to have students work aloud to you on column addition.

In column addition, it is necessary that the exercises be carefully designed so as to keep students away from unfamiliar combinations. Also, it is vital that some decision be made as to the direction of adding. For many years, all column addition was upward. Frequently, the suggestion was made that you check your result by adding down, but addition was upward. Recently, however, there has been pretty general agreement that downward addition is preferable. Several reasons are cited for this: (1) Downward addition yields the sum near where it is to be recorded. A student who adds upward must shift his attention from the top of the column to the bottom, keeping his answer "in his head." There is ample opportunity for the answer to get lost in this process. (2) Sweeping your eyes from top to bottom of a column is a more natural movement than sweeping them from bottom to top. (Try it and see!) (3) Since we teach that addition and subtraction are opposite processes, and since subtraction is normally upward, we can bring out the "oppositeness" by adding downward. (4) If someone reads a column of figures for you to add, wouldn't you write them downward, with the last figure at the bottom of the column? If you write them in that sequence, isn't it logical to add them the same way?

Even more important than arbitrarily setting an add-up or add-down rule, however, is the principle that we follow one pattern consistently, thereby minimizing confusion. Later, we shall suggest that the students check by adding in the direction opposite from that used at first.

A beginning fifth-grade teacher was given a class that had, throughout earlier years, been taught to add downward, then check by adding upward. The first time he had occasion to do a column addition, he added upward as he had always done. He became very uncomfortable when his work was greeted with laughter. Finally, he drew from the class the explanation that "You're checking it before you work it."

Two-place numbers. Students who are clear on the structure of numbers have suprisingly little trouble with adding two-place numbers

(without carrying, of course). Actually, wben properly introduced, this operation does not involve a single new concept.

Some texts present this type of addition by reverting to concrete objects briefly. Even if this is not done, it would be well to review witb the class the fact that 21 means 2 tens and a one. Also, the principle that numbers in the ones position can only be added to numbers in a ones position should be discussed. This, of course, was ioitially taugbt when addition was introduced.

With these basic points in mind, it is relatively easy to see that in

$$\begin{array}{r} 21 \\ +12 \\ \hline \end{array} \quad \text{we have} \quad \begin{array}{l} 2 \text{ tens } +1 \\ 1 \text{ ten } \;\; +2 \\ \hline 3 \text{ tens } +3 \end{array}$$

The student should be able to pass rather rapidly to

$$\begin{array}{r} 21 \\ +12 \\ \hline 33 \end{array}$$

It should be pointed out to them that we put these numbers in columns so that we can be sure we are adding ones to ones and tens to tens. A few students might need to go back to the place-value pockets in developing an understanding of this process, but they should be encouraged to move away from it as soon as possible. Remember that a teaching material can also be a crutch.

SUBTRACTION

Since the teaching of subtraction closely parallels the teaching of addition, many of the basic principles described io the previous section apply here. It is never safe to assume, however, that the second-grader is clear on the true meaning of subtraction—that is, the uo-grouping process. Consequently, the first task is to learn what needs to be retaught, then to proceed with the reteachiog operation.

Shift toward the abstract. Many second-grade teachers like to begin the work in subtraction with concrete objects, just as was true for first grade. Most students, however, should be able to pass to the abstract rather rapidly. A key point in using the concrete objects is that no student use them longer than is necessary. This principle applies to subtraction as well as to other operations.

You have noted that first-graders use a graphic term for un-grouping. They call it "take away." In most programs, second-graders are

introduced to the term *subtraction.* Also, the symbol for this operation is usually taught quite early in the school year.

New facts introduced. The first-grade program in subtraction frequently consists of the 15 easiest subtraction facts. In second grade, these facts are retaught, and the list is expanded considerably. Some programs include all of the subtraction facts with minuends of 10 or less; others include minuends of 12 or less; a few cover all the subtraction facts by the end of second grade. Many teachers feel that the minuend-of-10 program, including 45 subtraction facts, is too modest; even more feel that mastery of all 81 facts is too ambitious. Probably your program will include some number of facts between these extremes.

As was true in addition, the new facts are usually introduced pictorially, with the symbols accompanying the "story." The students move toward use of numerals without drawings as rapidly as possible.

Frequently, the teacher asks herself, "To what degree should I expect mastery of these facts?" Of course, all the subtraction facts are studied again in third grade. There can be no doubt, however, that the ultimate goal is complete mastery, mastery to the degree that a student looks at $\begin{array}{r}5\\ \underline{-2}\end{array}$ and thinks "3." One of the greatest difficulties in arithmetic is that some students lack such mastery, hence cannot proceed with confidence. This is not to say that such mastery must be achieved in second grade; however, definite progress in this direction is most desirable.

Certain arithmetic programs are varying the pattern on addition and subtraction facts to some degree. For example, there is a great deal of carry-over in learning $\begin{array}{r}5\\ \underline{+4}\end{array}$, $\begin{array}{r}4\\ \underline{+5}\end{array}$, $\begin{array}{r}9\\ \underline{-4}\end{array}$, and $\begin{array}{r}9\\ \underline{-5}\end{array}$. Hence, some programs are introducing such facts by families of this sort. This approach seems to have much to offer and is likely to come into more widespread usage.

Two-place numbers. Many second-grade programs are including subtraction of two-place numbers without borrowing. This, of course, does not represent anything particularly new or difficult to students who understand place value and who know the required subtraction facts. The exercise $\begin{array}{r}23\\ \underline{-12}\end{array}$ is usually presented, by way of review, as

$$\begin{array}{l}2 \text{ tens } +3 \text{ ones}\\ \underline{-1 \text{ ten } \ \ +2 \text{ ones}}\end{array}$$

Familiar principles immediately yield 1 ten + 1 one as the difference. Some students would need to verify results with concrete objects or even with the place-value pockets in the early phases of such work. Generally, a student who has mastered the necessary subtraction facts encounters little difficulty in subtraction of two-place numbers without borrowing.

Aids available. Many of the teaching materials that would be helpful in teaching addition and subtraction at the second-grade level have been described earlier—such as counters of various types, place-value pockets, films, and filmstrips. Several of the latter might be used only in part, the teacher selecting those sections which apply. This is especially true of filmstrips; it is not unusual for a teacher to use only a few frames from a filmstrip.

Any list of teaching materials rapidly ceases to be timely, since new materials are constantly being developed. Nevertheless, one of the best available discussions of teaching materials appeared in 1951.[1]

There are a number of misconceptions regarding the use of materials in teaching arithmetic. Certain teachers have cited a lack of materials as a reason for giving arithmetic rather casual treatment in the daily schedule. Since a complete collection of aids does not and cannot exist, any teacher has a ready-made alibi for a poor teaching job if she is looking for one. Teachers who have had long and successful experience in the use of teaching aids in arithmetic accept the fact that some of the most effective aids are available in any normal classroom. Certainly at second grade, such materials need not be sophisticated in order to be useful.

An attitude which parallels the first is that a material must be "store-bought" in order to be good. The commercial product may be more sturdy or more showy than the one the teacher could make, but it doesn't follow at all that one would be superior to the other as an aid in teaching. Further, wouldn't it be good teaching to give the students a chance to help in preparing such materials? Frequently, they can do a surprisingly good job on them. And there would be automatic interest in a device that Joe or Sally prepared.

Problem Solving. The most challenging part of any arithmetic program is the problem solving involved. A problem is generally thought

[1] Foster E. Grossnickle, Charlotte Junge, and William Metzner, "Instructional Materials for Teaching Arithmetic," *Fiftieth Yearbook of the National Society for the Study of Education* Part II (Chicago: University of Chicago Press, 1951).

of as a verbal statement (some books call them *stories*) of a situation requiring a solution, with no statement given as to how to proceed. Hence, the student must (1) decide what operation to use, and (2) carry out the operation. This, you will note, is considerably more difficult than a series of exercises where the directions plainly tell the student what to do.

At second-grade level, extensive use is made of problem solving. The situations described are such as to be of interest to most second-graders. Consider this example: "Tom brought 3 turtles to school on Monday. On Tuesday, Bobby brought 4 turtles. How many turtles did they have at school?" On the same page are some problems requiring subtraction. Hence, the student would have to read the problem carefully, decide what operation to use, then carry out this operation.

A common error in problem work is relying solely on texts and workbooks for problems. Every classroom offers the teacher numerous opportunities for problem work. Even using the names of members of your class in setting up problems will add interest and reality. Also, we can always call upon students to set up problems for the class. Even second-graders can be quite ingenious at this. Some are better at making problems than at solving them.

The whole area of problem solving is complex, and it gives a great deal of trouble. Many studies of it have been made, but no panaceas are in sight.

MULTIPLICATION AND DIVISION

A few arithmetic programs introduce the operations of multiplication and division in third grade. It is a more common practice to start them in second grade. The emphasis is placed on the meaning of the operations, with little effort to teach an extensive list of multiplication and division facts.

Grouping again. The importance of seeing groups in addition and subtraction has been emphasized. The ability to think in terms of groups is equally important in the other two operations.

The introduction of multiplication is usually carried out without any use of the word. Rather, it is presented as a logical next step based upon two earlier operations, group counting and addition. One approach that might be used is to have a student count by twos to 6 or 8. As he says "two," the teacher draws two circles on the board. As he counts upward, she draws another pair of circles, then another. It is important

that these be drawn in such a way as to bring out the "twoness." Hence, the pairs of circles should be clearly shown. The students would be shown that each group consists of "a two." This type of thinking as applied to tens has already been mentioned.

By gradual development, with much discussion and possibly with some references to the concrete, students come to see the total group of six as consisting of a two, a two, and a two. From this point, it is an easy step to show that the group consists of 3 twos, or 6. Sweeping separate groups of objects into a single group dramatically shows the process of multiplication.

The terms we use in this operation have changed considerably. In earlier years, the process might be described as "threetum two is six." More correctly, of course, it would be "three times two is six." Presumably this initially meant that if we took 2, three times, we would have 6. The abbreviated version, "three times two," fails to describe what actually occurs. But when we say "3 twos," we are describing what occurs. Objects are arranged into twos, and we are considering three such pairs. This terminology has been troublesome to some teachers who learned the "times" system, but the change has contributed clarity to the process. Later, of course, there are occasions when the *times* term will be used; for example, in common fractions.

Another approach to multiplication, frequently combined with the one just described, uses addition as the starting point. Addition, you will recall, is a procedure by which two or more groups are combined into a single group. The student can readily see that we get the same end result in multiplication; that is, separate groups are combined into a single group. He should also discover, however, that one limitation applies here that was not present in addition; namely, the separate groups in multiplication are equal in size. Thus it is seen that multiplication is a special case of addition in which equal groups are combined. The writers recall an eighth-grade student who learned this lesson so well that she didn't want to unlearn it. If she had occasion to use $\begin{array}{r} 8 \\ \times 7 \\ \hline \end{array}$, she would put down 56 as the product. Then, in the margin, she would list 8's (seven of them) in a column and add them as a check. She lacked confidence in her knowledge of one of the basic facts.

In many programs of arithmetic, the processes of multiplication and division are introduced almost simultaneously, since the two processes are closely related. Again, the term *division* is not used. Rather, the

student takes a larger group, such as the six used earlier, and sees how many smaller groups can be made from the total number of objects or pictures.

Using the earlier illustration, you might take six pencils as an initial group. The question would be "How many twos are in this group?" The students would then rearrange, or regroup the pencils, arranging them in twos. Of course, three such groups would result. Thus, the students have answered a basic question. "How many twos are in six?"

The students are likely to see a relationship between division and subtraction. The latter is also a regrouping operation. In division, however, a limitation is functioning in that we subtracted twos and only twos from the original six. Hence, we have in effect subtracted equal groups from six until nothing remains. So division is actually a special, and more rapid, form of subtraction involving removal of equal groups.

Again, the terminology for this process has changed. For many years, the usual statement was, "How many times does two go into six?" or "Two goes into six how many times?" Neither term was very descriptive, since it was apparent to the student that the two didn't *go* anywhere. It was still very much in evidence after the exercise was completed. Many students even combined the *goes into* term into a single word sounding like "guzinta" which is even less descriptive of the operation. In the introductory phases of this operation, most students now use the question, "How many twos in six?" This actually tells what is to be done. The original group is to be broken into smaller groups, and we determine how many such groups result.

Scope. Many of us recall that our first major contact with multiplication came when we reached the "multiplication tables," an awesome device showing all the facts through 12 × 12. Some teachers took the inflexible position that "Here we stay until all the students can say all the tables." Now that research has indicated that we learn better with repeated contact (rather than a prolonged single period of study), most of the basic operations are studied on a fairly limited basis each year for several different grades.

As mentioned earlier, some books do not introduce multiplication and division in second grade. Generally, those that introduce these processes place the emphasis on understanding the principles involved. This, of course, requires that certain of the facts be studied.

Some programs take a few facts and study both multiplication and

division by "families." Typically, activities might be built up around groups like

$$\begin{array}{r}2\\ \times 3,\\ \hline 6\end{array}\quad \begin{array}{r}3\\ \times 2,\\ \hline 6\end{array}\quad \begin{array}{r}3\\ 2\overline{)6},\\ \end{array}\quad \text{and}\quad \begin{array}{r}2\\ 3\overline{)6}\\ \end{array}$$

The same might be done for 8 or 10 as products or dividends. Little effort, however, is made to try to have the students achieve mastery of any particular groups of facts in multiplication or division during second grade.

Problem solving. The effort to develop skills in solving problems begins early and continues all the way through arithmetic. Textbooks frequently use simple problem situations in introducing processes such as multiplication and division. Some activities that are frequently used for problem work are the lunchroom line; the lunchroom table; cakes, fruit, or candies in a variety of groups; tops, blocks, or coins in various patterns. In each case, real problems are presented.

The alert teacher will find, however, that her best problem situations occur, either accidentally or by careful prearrangement, in her classroom. Certainly, these problem situations would be real, hence interesting, to her students.

One basic change in the approach to problem work is noted. For many years, students studied a process chiefly by memorizing facts and then, having mastered enough facts, they moved into problem work as an application. Now, problem solving is widely used in introducing and building understanding in various processes. Actually, in the modern arithmetic program, we usually build from, rather than toward, applications.

Aids available. The most effective teaching materials in multiplication and division for second grade are the concrete materials that have been used for other purposes. Of course, there is opportunity for variety here. A boy would probably work more effectively with grouping or ungrouping marbles or balls than with dolls. Girls would probably be more interested if this pattern were reversed.

Most of the films dealing with these processes are too advanced for second grade, but there are several filmstrips of which some parts could be used. Some of these are

1. *Using and Understanding Multiplication*
2. *Using and Understanding Division*

3. *Multiplication and Division*
4. *The Twos in Division*
   (All of the foregoing are available from Stanley Bowmar Company, Valhalla, New York)
5. *Multiplication and Division*
   (by Young America Films, Inc.)

Semiquantitative terms. In first grade, several terms of a semiquantitative type were studied. At second-grade level, most of these are retaught. Several additional comparison terms are usually introduced at second-grade level. Some typical examples are low, lower, lowest; small, smaller, smallest; much, more, most; many, few; bottom, top; over, under; and high, higher, highest. This list, of course, varies from one text to another.

One occasionally hears the argument that attention to the semiquantitative terms is not a part of arithmetic, since no number manipulation is involved. Such terms, however, are a part of the everyday experiences of children. If these words are to convey meaning rather than confusion, it is essential that they be studied by the class. Many teachers are now giving a relatively brief treatment of them in lower elementary arithmetic classes.

FRACTIONS

For many years, the study of fractions was reserved for the upper elementary grades, presumably because some of the processes involving fractions are fairly complex. The concept of fractions is basically simple, however, and is frequently introduced in first grade.

One-half, one-third, one-fourth. The term *fraction,* based upon the idea of "fracture," or breaking something in two, usually is not introduced when the idea of fractions first appears. Many first-grade programs are now introducing one-half of a single object. Of course, full use is made of the fact that for years the students have heard references to half-dollar, half an hour, and "meeting you half-way."

Even at first-grade level, it is important that students understand the *condition* that must be met in producing one-half, namely, that the original unit be broken into *two equal* parts. One first-grade girl very carefully broke a stick of gum into three pieces, then passed them to her two friends, saying, "Here's half for Mary, half for Sally, and half for me."

The work in fractions at second-grade level includes the reteaching of one-half. The usual next step is to go to fourths, since dividing a half into two equal parts is relatively easy. Usually, both halves will be divided so as to show the four parts which provide the base for the term *fourth*. This phase of work is not presented as anything new or different.

Although some programs go into one-third at second-grade level, this is by no means general. Indeed, one might argue for taking up one-eighth before one-third.

Note that, at this grade level, only unit fractions are introduced, that is, fractions with a numerator of 1. No attempt is made to show that two-fourths equal one-half, and no processes are dealt with. The only goal is introducing the concept of fractions in a way that will have meaning for small children.

Some limitations. It would probably be very confusing to children if they had to study fractions by using number language, since the symbols do not relate to anything they have previously studied. Consequently, in first and second grades, the symbols are not ordinarily used. Rather, the words "one-half" and "one-fourth" are used. This doesn't present any particular problem, since no manipulation is taught at this level. Also, the relationship between "four" and "fourth" shows very clearly in the word usage.

In the fraction work at second-grade level, the concrete is used extensively. Apples are cut in half, sheets of paper are cut into halves and fourths, and many other such activities are carried out. The texts and workbooks also present semiconcrete pictorial material. Somehow, cakes and pies get great emphasis here, frequently accompanied by fruits, candy, and other foods. Common practice is to apply the terms *one-half* and *one-fourth* to a single object rather than to a group. It is much easier for a child to visualize half of an apple than half a dozen apples. The later, of course, implies the ability to "see" a group as a unified whole, a skill which frequently has not been developed by second grade.

Aids available. We have already mentioned some of the most effective aids for use in teaching fractions, such as apples or other objects that are readily divided. Sheets of paper can be quite effective here, also. A group might be challenged to see how many different ways they could divide sheets into halves, then fourths.

Teachers who prefer to use "store-bought" aids would probably like

such materials as the "Fruit Plate," sold by Creative Playthings, Inc., New York. This consists of a pear that is cut into halves, an apple in thirds, and an orange in fourths. These are sturdily built and can be assembled and disassembled many times over without showing appreciable wear.

Another popular aid is the rubber fraction pie, available from several different distributors. Pies that are cut into halves, thirds, and fourths are available, the sections being large enough for easy manipulation. A variation of this is the flannel board with fractional parts, like that sold by the John C. Winston Company, Philadelphia.

A few filmstrips might be adapted, at least in part, for use at this grade level. For example, certain frames of "Meaning of Fractions," by Young America Films, or "What is a Fraction," by Filmstrip House, might be used. Most of these aids, however, are designed for use with older children.

MEASURES

As students advance through the grades, their study of measurement undergoes two types of changes: (1) new units are introduced; (2) smaller subunits are taken up, the goal being to achieve more accurate measurements. Many second-grade programs place the emphasis on smaller subunits and do not introduce any new types of units.

Money. Second-grade programs expand upon the first-grade study of money by introducing some new coins. Indeed, it is not unusual to find second-graders working with pennies, nickels, dimes, and quarters, and the half dollar is sometimes included.

A variety of activities is included in this work. For example, attention is usually given to such questions as, "How many pennies equal a nickel?" or "How many pennies equal a dime?" Also, the fact that two nickels equal a dime, or that five pennies and a nickel equal a dime might be included.

The counting of money, involving a mixture of coins, nickels, and pennies, for instance, is usually taught. Another fairly typical activity is simple addition and subtraction problems with either real or pictured coins. Problem solving, sometimes requiring use of all four of the fundamental operations as applied to money, usually gets some attention.

Some programs give second-graders limited contact with the process of making change. This usually follows the study of converting

quantities from one type of coin to another. Change-making, however, is not usually emphasized at this grade level.

Time. The study of the clock dial usually begins in first grade. Manipulative work is carried out, the emphasis being on reading time to the hour. Many second-grade programs reteach this and expand upon it to include the half-hour. This is usually rather slow since, after all, the system we use is relatively complex. Most of the work with time is based upon the same types of activities described for first grade. There is likely to be more emphasis on pictorial material, with less use of manipulative aids than was true in first grade. Texts that include the half-hour for second grade usually minimize confusion by showing dials with readings on the hour or half-hour, with no intermediate readings.

Distance. The distance concept, of course, must be retaught. Many second-grade texts make use of the foot and the inch as units. The usual types of activities are included, all of them built around actual measuring situations. The ordinary classroom offers an endless variety of opportunities for this type of work.

Attention is given to the fact that there are 12 inches in a foot, and some work in converting from one of these units to the other may be included. Many teachers like to use rulers that have no subdivisions smaller than an inch. These are available from most of the educational supply houses.

Some programs include a few other types of measures at second-grade level. The *dozen* is sometimes used. A few give some work in simple volume measures, such as the pint and quart. Some include weight in pounds. Most texts, however, limit themselves to measures of time, money, and distance.

Aids available. For working with money, if the teacher is sure that the students recognize real coins, play money may be used. Frequently, a dime store or school-supply store will have play coins that are quite realistic in appearance. If, however, there are some students who are not sure they can tell a nickel from a penny or a dime, it is best that they make further use of real coins.

For the work with time, the students can often bring inoperative alarm clocks from home. Several companies have clock faces available at a nominal price. For example, the John C. Winston Company sells a very sturdy and durable model. This company also sells another

device that would be most helpful to the teacher in preparing worksheets for class use. This is a rubber stamp of a clock face, without hands, of a size that permits six faces on an 8½- × 11-inch sheet of paper. This could save a great deal of teacher time, as she could easily sketch in the hands to show a particular time or have the students draw in the hands for specified time readings.

Distance measures require a ruler of some sort. In some areas, certain companies distributed rulers or yardsticks as advertisements. If the teacher does not object to a bit of commercialism, these are readily usable aids. If some of the students find the subdivisions (usually to 1/16 inch) confusing, the teacher may be able to use masking tape to cover the space between the inch markers. Alternatively, if she can get some small strips of wood, they can easily be cut to 12-inch lengths and the inch markers drawn on them. And, as mentioned earlier, rulers showing only the inch markers are commercially available.

There are a few filmstrips that might be of assistance. For example, Filmstrip House has one entitled, "Man and Measures," which gives some interesting bits of history about early methods of counting and measuring. Only certain parts, however, would be usable at second-grade level.

C. The Slow Learner

The wide range of ability is with us at *all* grade levels. By second grade, all students are likely to have made the adjustment to school life. Many of the problems described in the previous chapter as applying primarily to first grade will have been reduced or overcome. But there is no formula that will eliminate the necessity of working with a range of abilities. Homogeneous grouping? The only homogeneous group is a "group" consisting of one person.

Actually, the range in innate ability increases as students advance through the grades. A six-year old with an IQ of 70 has a mental age of 4.2 years; another six-year-old with an IQ of 140 would have a mental age of 8.4 years. The difference would be 4.2 years. If, however, we consider these same two students as twelve-year-olds, assuming the IQ remains constant, the difference between their mental ages would be 8.4 years. Consequently, the range in mental ages steadily increases as students become older.

The list on page 122 gives some practices that are commonly recommended in working with the slow learner. The teacher will doubtless develop others that are suited to her own group.

1. The level of achievement must be differentiated. Some students, despite a valiant effort, will fail to achieve as the teacher would wish them to do. But by keeping the slow learner working on the essentials, it is hoped that he will be able to master selected basic ideas of arithmetic.
2. The slow learner will require far greater use of the concrete than will the average and above-average student.
3. Whenever feasible, a variety of aids should be used, since each new approach makes its own contribution to learning.
4. Taking a lesson from the teaching machine approach, new topics should be broken down into minute steps whenever possible.
5. A great deal of drill is necessary for the slow learner. If this can be varied so as to add interest, it should be done.
6. A large amount of reteaching of fundamentals should be used. Unfortunately, the slow learner is "a fast forgetter."
7. Look for opportunities to let the slow learner experience some success. If there is anything he can do well, give him an opportunity to display this talent.
8. Constant attention to diagnosis is necessary, and this applies to a wide range of arithmetic activities. Along with this, the teacher should observe constantly to be sure that the slow learner is not developing work habits that will further handicap him.

It should be pointed out that there are rewards to the teacher who is willing and able to deal successfully with the slow learner. A veteran teacher once remarked, near the end of a forty-year teaching career, that most of his lifelong friends among former students came from the average and below-average ability groups. He said that he never wrote a student off as a hopeless case but tried to contribute to the growth of each, regardless of ability.

D. The Rapid Learner

In some respects, the rapid learner is harder to work with in a class than is the slow learner. The rapid learner finishes work that would require a considerable amount of time for average or below-average students almost by the time the teacher has completed the assignment. This gives an alert second-grader a golden opportunity to get into trouble. The best course of action for the teacher is to have supplemental work ready for such an occasion.

IDENTIFYING THE RAPID LEARNER

Many schools use a battery of achievement tests early in the school year. A student who performs at an exceptionally high level on these tests should be observed closely to see whether this type of work is characteristic. If the school doesn't give an achievement battery in the fall, the teacher may need to review the cumulative folders to look for evidences of acceleration. Probably the best single source of information, however, is the teacher who had the class the previous year. Although contemporary teachers no longer worship at the shrine of the IQ, results of intelligence tests often contribute to our understanding of a youngster's potential.

It is not unusual for the teacher to disagree with a parent regarding a child's precocity. Frequently, a second-grader is encountered who gives evidence of being a rapid learner because of special types of out-of-school opportunities and home background. One might, for example, have a second-grader who is well above grade level in his knowledge of the addition or subtraction facts. Often, this merely indicates that parents have drilled him on these facts. If he is a rapid learner in arithmetic, it usually shows up in such areas as ability to generalize, ability to see relationships, and ability to see problem situations. In the final analysis, the rapid learner is identified by an observant teacher. Other techniques help, but teacher judgment is the final criterion.

CHALLENGING THE RAPID LEARNER

One way to challenge the rapid learner in a second-grade arithmetic class is to have him work at third-, fourth-, and fifth-grade level. If there were no other way, this would doubtless be used. But the surest way for a second-grade teacher to make a negative impression on her colleagues is to let the word get out that she is "infringing on the subject matter" of a later grade. Hence, the emphasis is on horizontal enrichment, that is, having the student study in greater depth those topics that are normally a part of second-grade arithmetic.

Some teachers' manuals are very helpful in this respect. For example, the teachers' edition of *Making Sure of Arithmetic, Book Two*,[2] has a regularly recurring topic on meeting individual needs. One subtopic

[2] Robert Lee Morton and Merle Gray, *Making Sure of Arithmetic, Book Two—Teacher's Edition* (Dallas. Silver Burdett Company, 1958), p. 36 T and others.

gives good suggestions for enrichment and challenge for the rapid learner.

The following activities are frequently used for enrichment: preparing reports and scrapbooks (when kept above the busy-work level), devising new ways of doing things, and working on number recreation projects. Some teachers like to use rapid learners, even in second grade, as tutors. This device, if used at all, should be employed with great caution. The tutor may feel overly superior and the child receiving help may acquire feelings of inferiority.

It would be impossible to list all the enrichment activities available to a second-grade teacher, since many of them grow logically out of classroom activities, but a few good reference books may be valuable. One such book is by Spitzer.[3] This book has an excellent feature, an index which lists the projects or demonstrations along with the grade level or levels for which each activity would be appropriate.

Something to Think About

1. In your opinion, is the present-day emphasis on understanding the processes of arithmetic justified? Can you cite evidence to support your position?

2. If you were an elementary principal, how would you respond to the complaint of a second-grade teacher who says that she cannot teach place value because the school will not buy place-value pockets for her to use?

3. Examine a diagnostic test and a readiness test in arithmetic. How do they differ as to (a) structure, (b) method for use?

4. One sometimes hears a student say that he has a "mental block" regarding mathematics. Can you cite evidence that would confirm or refute this statement?

5. Can you find research reports dealing with the teaching of addition? Why not give the class a summary of one or more of them?

6. Does it really matter whether a student adds upward or downward in column addition? Upon what basis did you arrive at your answer?

[3] Herbert F. Spitzer, *Practical Classroom Procedures for Enriching Arithmetic* (St. Louis: Webster Publishing Co., 1956), pp. 217–24.

7. What provision would you make for a second-grade student who actually does not need the extensive reteaching usually done at this grade level?

8. How would you explain to a parent that you say "2 fours are eight" rather than "two times four are eight"?

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Arithmetic in Grade Three

There is no such thing as *the* arithmetic program for grade three; there is such a thing as *an* arithmetic program for a particular grade. There is no such thing as an average program. We may find that one author includes the process of carrying in second grade; another introduces carrying in third grade. How could these be averaged? The program found in each arithmetic text series is based upon the author's concept (usually supported by research) of what should be taught and when.

Beginning third-graders usually have been "exposed" to about one-half of the addition and subtraction facts. The basic principles of multiplication and division have been introduced. Yet it is never safe practice to assume that these have been mastered. A certain amount of time, depending upon the needs of the class, should be devoted to reteaching each process as it is reintroduced to third-graders.

Topics presented in this chapter are

A. Introductory Activities

B. A Third-grade Arithmetic Program

C. The Slow Learner

D. The Rapid Learner

A. Introductory Activities

Of course, the first job of the teacher as she begins the year with her new third grade is to get to know her class. She starts with the information in the cumulative folders of students and the oral opinions of the second-grade teachers. In many schools, a battery of achievement tests is given early in the year, since this is another device to help the teacher know more about her class. If such a test is used, it should be delayed until the usual beginning-of-the-term turmoil is over; that is, until the students have adjusted to the routine of school life and have had some renewed contact with arithmetic. Although many of them have done some reading during their summer vacation, there is little likelihood that they have worked any arithmetic.

Achievement tests. There is no general agreement as to the best time of the year to give standardized achievement tests. It is likely that your school will have an established program of testing, along with a schedule. There can be little doubt, however, that, if the test results are to be of maximum help to you, the third-grade teacher, the tests should be given fairly early in the school year.

Should you have the responsibility of selecting the arithmetic test or tests for use with your class, there are several sources of information to which you may turn. The standard reference in such matters is Buros, *Mental Measurements Yearbook*, published by Rutgers University Press. A more specialized reference is Myers, *Mathematics Tests Available in the United States*, published by the National Council of Teachers of Mathematics. This book lists about fifty standardized tests, achievement and otherwise, in arithmetic. The catalogs of the various test publishers can also be quite helpful. Before making a final selection, the teacher should order specimen sets of the more promising tests. Since these sets are relatively inexpensive, the teacher can examine the test itself.

In schools where an achievement battery is used, most teachers depend upon the arithmetic section of this group of tests instead of using a separate test. Some of the more popular batteries are *Stanford Achievement Tests*; *Metropolitan Achievement Tests*, both published by Harcourt, Brace and World; and the *California Achievement Tests*, published by the California Test Bureau. If there is need for doing so, teachers can order the arithmetic sections of these batteries, without buying the entire set.

Diagnostic tests. Diagnosis for difficulties in arithmetic is an activity that continues throughout the year. Although diagnosis is, in part, a matter of teacher attitude, there are several aids available. As was mentioned earlier, many textbooks have diagnostic tests at the end of each section, and these can be very helpful.

Also, there are some standardized tests available that are especially designed to help the teacher diagnose learning problems. Some of these are

1. Buswell, John, "Diagnostic Tests for the Fundamental Processes in Arithmetic" (Grades 2–8, Public School Publishing Company)
2. Brueckner, "Diagnostic Tests and Self-Helps in Arithmetic" (Grades 3–12; California Test Bureau)
3. Armstrong and Clark, "Los Angeles Diagnostic Tests: Fundamentals of Arithmetic" (Grades 2–8; California Test Bureau)
4. Armstrong has a test similar to the foregoing in reasoning in arithmetic (Grades 3–9; California Test Bureau)

Some of these tests are quite old, but they still can be effective in helping teachers spot specific learning difficulties in arithmetic.

B. A Third-grade Arithmetic *Program*

As will be seen, several new number concepts are usually introduced in third grade. Considerable time is used in reteaching. Probably the most time-consuming phase, however, is the spiral expansion of concepts that have been introduced in previous grades.

NUMBER SYSTEM

Most of the work in grades one and two was based on numbers expressed by one- and two-place numerals. Third-graders usually study numbers up to 1000 or above in reading, writing, counting, and using in operations. Considerable attention is given to numbers in the hundreds. Working concretely with numbers in the hundreds can become very cumbersome and time-consuming. One helpful technique makes use of Christmas seals or other such materials. These are normally mailed out in sheets of 100, usually 10 by 10. Here the fact that there are 100 objects present is immediately apparent, yet there is a minimum of counting.

PLACE HOLDER AND PLACE VALUE

Some texts give an extra bit of attention at third grade to the role of the zero in a numeral. Actually, numerals in which zero appears have been used all along, but usually zero was treated simply as a part of the number symbol. Now, with the aid of pictorial material, the student is shown that, in the numeral 201, for example, the zero means "no tens." Many third-grade texts introduce the term *place holder*, which describes the function of the zero, to the students at this time.

It should be pointed out, however, that misconceptions can arise regarding the place-holder principle. One that can give trouble is the glib statement, "Zero is a place holder." Some students interpret this to mean that zero is not a number, just a place holder. Zero is a number. Like other numbers, it has several functions. One of these is as a place holder.

Obviously, the term *place holder* would be meaningless to a student unless he was clear on place value. Hence, a review of the principle of place value is essential. This time, the hundreds position on the place-value pockets is involved. Most teachers and students, however, find it tedious to do very much concrete work on this. For example, considerable time would be used in bundling ones to tens, then tens to hundreds. Some teachers avoid this by making the ones pocket, the tens pocket, and the hundreds pocket of different colors, say red, green, and blue (Fig. 5.1). When the number of sticks in the ones pocket reaches 10, they are bundled as usual, but the bundle is represented by a single green stick in the tens pocket. Likewise, 10 tens would be bundled and represented by a single blue stick in the hundreds pocket. A few texts recommend this procedure in the introduction of place value, but it would probably produce some confusion in the minds of those hearing of place value for the first time.

Counting by groups. Because of the continued emphasis on grouping as a vital technique in understanding numbers, teachers should give

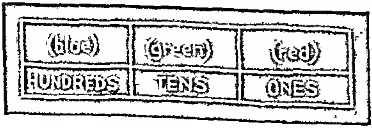

Figure 5.1

third-graders a considerable amount of work in group counting. Counting by tens, fives, and twos was introduced earlier. Many programs expand this to include counting by threes and fours. Frequently, the student might be called upon to count by groups, beginning at values other than zero, such as counting by fours starting with 6.

Roman numerals. Several third-grade texts introduce Roman numerals, usually going up to XII. Probably this is done because the clock dial is frequently used as a starting point. Some teachers question the value of teaching Roman numerals at all because they are not functional in our society. True, except in outlines, on clock dials, cornerstones, and a few other instances, we seldom encounter this type of numeral. Consequently, the study of Roman numerals is usually presented as a topic of historical interest only, and such work does not receive major emphasis.

Teachers are frequently bombarded with questions when the Roman system is being studied. One such question has to do with the merits and weaknesses of the Roman system. Several of the weaknesses are immediately apparent, such as the fact that the Roman numeral system has no zero. Hence, one of the most vital symbols is missing, and there is no way of indicating a place holder. Another weakness is that the principle of place value is used only in a very rudimentary way. This is seen in such numerals as IV, where the I is subtracted, and in VI, where the I is added. Because of these two weaknesses, computation is extremely cumbersome with Roman numerals.

If this system of numerals has a strength, it lies in the fact that numbers can be written with so few characters. Our decimal system requires ten different symbols to write to 12, whereas the Roman system requires only three (I, V, X).

Some teachers find the Roman system to be a good source of research assignments for rapid learners. In general, however, this topic receives only casual treatment at third-grade level.

Addition and subtraction. One pattern regarding addition facts is to aim for mastery of those facts involving sums up to, and including, 6 in first grade. In second grade, these facts are retaught and the list expanded to include sums to 10 or 12. Then, in third grade, all of these facts are retaught, and the list is expanded to include all of the 81 facts (excluding zero). Thus it is hoped that, by the end of third grade, all the addition facts will have been mastered. As has been true in earlier grades, we keep the addition facts and the subtraction facts closely

coordinated. Hence, by the end of third grade, the 81 subtraction facts will normally have been taught.

After the students have been introduced to all the addition and subtraction facts, achieving mastery is given a great deal of emphasis. A wide variety of techniques is used, such as flash cards, various types of games, phonograph records, and others. All of these represent ways of adding some degree of enjoyment to the vital process of drill. But however it is handled, drill is necessary if students are to achieve the goal of mastery.

Some teachers find it easier to improvise exercises after all the facts have been introduced, since the bothersome question, "Have they had this one?" does not apply. Certain teachers, however, tend to overstress favored facts while almost excluding others. This same pattern is observed in a few texts. Hence, it is important that a teacher use some sort of guide or pattern in preparing exercises in order to assure proper distribution.

Teachers sometimes become discouraged with the slow progress their pupils make in the task of learning the various facts. They should remember, however, that students are being asked to do something which is quite new and different to them. In reading, they try to learn to recognize printed or written symbols. But recall that, in an addition fact, they are asked to learn to recognize a sum that is not visible at all. Further, they are asked to learn these facts with no tolerance for error; that is, there is no "nearly right." This is unique in the lives of third-graders, so progress is usually slow.

Bridging. In column addition, it is frequently necessary for a third-grade student to use bridging, that is, to make the transition from one decade to another within the column. For example, consider the following:

$$\begin{array}{r} 7 \\ 8 \\ +9 \\ \hline \end{array}$$

The first operation, $7 + 8$, is one of the addition facts and would normally give little trouble. But this unwritten (hence easily forgotten) 15 is then added to 9. This requires that the student bridge from the tens to the twenties, and he may find this confusing.

One approach is to give the students extra drill on the similarity between

$$\begin{array}{r} 15 \\ +9 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{r} 5 \\ +9 \\ \hline \end{array}$$

Indeed, a few authorities have urged that students learn as addition facts all of those now included plus some upper decades of them. This would mean that a student would learn

$$\begin{array}{r}5\\+9,\\\hline 14\end{array}\quad\text{then}\quad\begin{array}{r}15\\+9,\\\hline 24\end{array}\quad\text{then}\quad\begin{array}{r}25\\+9\\\hline 34\end{array}$$

Most teachers, however, would not relish the job of teaching several hundred addition facts, in view of the troubles encountered in teaching the usual 81 facts.

Another approach is use of the "number line." This is usually a strip of paper or cardboard mounted on the wall with the numbers 1 to 50, or 1 to 100, equally spaced on it. If such a device is available, it is relatively easy to show the similarity among

$$\begin{array}{r}5\\+9,\\\hline 14\end{array}\quad\begin{array}{r}15\\+9,\\\hline 24\end{array}\quad\text{and}\quad\begin{array}{r}25\\+9\\\hline 34\end{array}$$

Incidentally, the number line is described by some authors as a very versatile and useful device in teaching several different operations.

At the time that bridging is being taught, the teacher should be on the alert for faulty work habits. These are much easier to prevent than to correct. One habit that easily creeps in at this point is the tendency to engage in an extensive soliloquy, or to talk through the operation slowly and carefully. This is usually a stall for time and indicates a lack of self-confidence in the operations involved. More drill on this phase of addition frequently corrects such difficulties.

Carrying. Some teachers get confused on the difference between bridging and carrying. The former occurs *within* a column; the latter involves a transition from one column to another.

To certain students who have, with difficulty, mastered the facts and then struggled through bridging, it comes as a blow when they realize that there is still more, namely, carrying. Some teachers like to lead into this process by showing a need for it. The authors observed a teacher as she led her class through some preliminary work in addition with bridging. Then, with no indication that anything new was involved, she put $\begin{array}{r}18\\+\ 13\\\hline\end{array}$ on the board. The class floundered briefly, then recognized that here was an example on which their procedures would

not work. Now, the class, having realized the need for something new, was ready to explore the process of carrying.

One would not have to go very far back into the history of arithmetic to find this procedure:

$$\begin{array}{r} 18 \\ +13 \\ \hline 11 \\ 20 \\ \hline 31 \end{array}$$

The line of thought is 8 and 3 are 11, so write this down, with special attention to place value. Now we know that the ones in the addends represent 10 each, because of their positions. Hence we add 10 + 10 and place the sum as shown, again taking special care to position the digits properly. This method is sometimes shown for historical interest, but it is seldom used as a procedure to be applied.

Another approach is rewriting each of the addends to show the number structure. For example, in the illustration just given, we could write

$$\begin{array}{r} \text{1 ten 8 ones} \\ +\text{1 ten 3 ones} \\ \hline \text{2 tens 11 ones} \end{array}$$

Now, we know that 11 ones means 1 ten and 1 one. Therefore, the ten is combined with the other tens to yield 3 tens. Hence, our sum is 31.

A widely used method of teaching carrying goes back to the place-value pockets and counters. In presenting the example shown earlier, one student could place 1 ten and 8 ones in the appropriate pockets to represent 18. Another student could place 1 ten and 3 ones in the pockets to represent 13. At that point, the teacher might hope that someone in the class would object to the overpopulation of the ones pocket. Should all else fail, some judicious questions could be used to secure the objection. The class would probably propose, as a solution to the dilemma, that a ten be removed from the ones pocket. These 10 counters would be removed, grouped into a "ten" with a rubber band, then literally "carried" to the tens position. Since this procedure is based on materials and general principles that have been used repeatedly for other purposes, it is usually quite effective in introducing the process of carrying. As usual, the concrete materials are left out of the process as soon as the class can deal with it abstractly.

For years, authorities have differed as to whether or not students should write the carry number. For example,

$$\begin{array}{r} \text{①} \\ 18 \\ +13 \\ \hline 31 \end{array}$$

We know that many people write this number as a matter of habit and commonly get the correct answer. Some writers say that the writing of the carry number is a crutch and that it should never be done. Others recommend that this number be written as a "visual aid" in the process of understanding the carrying process. They usually suggest that the practice of writing the carry number be dropped as soon as possible. Of course, it is an open question as to how willing students will be to stop writing it after they have started. A veteran teacher ended a discussion on this topic with the remark, "If they can do without writing it down, they do not write it; if they cannot, they write it." This is probably as defensible a position as any.

Borrowing. Since most of the principles used in borrowing are similar to the ones used in carrying, these two operations are usually studied fairly close together.

Several different approaches have been used in teaching subtraction with borrowing during the past few decades. One of these, formerly more popular than now, was the equal-additions method. This made use of the principle, easily verified, that if you increase the subtrahend and minuend by the same amount the difference is unchanged. Consequently, in an exercise such as $\begin{array}{r} 41 \\ -29 \\ \hline \end{array}$ we could increase both 41 and 29 by 10 without changing the result. This increase would take place thus: to the 41, we would add *10 ones* to yield 4 tens 11 ones; to the 29, we would add *1 ten* to yield 3 tens 9 ones. The problem would then be reduced to

$$\begin{array}{rl} 4 \text{ tens} & 11 \text{ ones} \\ -3 \text{ tens} & 9 \text{ ones} \\ \hline 1 \text{ ten} & 2 \text{ ones} = 12 \end{array}$$

The key point to note in this method is that the amount added to the minuend is added as 10 ones, whereas the addition to the subtrahend is as 1 ten.

The borrowing operation is now more commonly taught by the decomposition method. This would mean that in the exercise just shown, the 41 would be decomposed to 3 tens 11 ones. The 29 can be subtracted from the 41 after the latter has been decomposed as shown. Some students will object that you cannot have 11 ones, since it violates all that they have been taught regarding number structure. Usually, however, the student will agree to the arrangement just shown on a "temporary" basis.

Some students would find it helpful to use place-value pockets in the borrowing operation, at least for the first contact with the process. Here the youngster is able to verify concretely that the borrowing is based upon the principles of number structure that are familiar to him.

Several arithmetic texts encourage the student to write the decomposition of the minuend. Thus, in the example $\begin{array}{r} 41 \\ -29 \\ \hline \end{array}$ the student writes 11 above the 1 and writes 3 above the 4, then proceeds to subtract. There can be no doubt that this will make the operation easier, but it brings up the age-old question, "Is it a crutch?" Those books that present such a procedure suggest that students discontinue its use as soon as possible. Others ask whether they will ever be ready to stop using this device once they have learned it. Generally, we should ask students to handle as much of a process mentally as they possibly can. Hence, it might be well to reserve such methods for those who require them.

A change in nomenclature seems to be under way regarding the term *borrowing*. Even third-graders have had occasion to learn that when you borrow, you must pay back. Yet any paying back in this process is on a contrived basis. Hence, you would look in vain for the word "borrow" in some of the newer texts, because the authors think that this term does not describe what occurs. In lieu of "borrow", the emphasis is being placed on the fact that a number is decomposed or rewritten in a different but equivalent form.

Checking. Since arithmetic by its very nature requires accuracy, the need for checking addition and subtraction exercises frequently arises in third grade. For checking addition, the most common pattern is adding downward to get the sum, then adding upward to check it. Some authors suggest that the student cover his sum with a sheet of paper so he will not be "prejudiced" to get the same result in checking.

It should be noted that in all cases except doubles, such as

$$\begin{array}{r} 3 \\ +3 \\ \hline 6 \end{array} \quad \text{and} \quad \begin{array}{r} 4 \\ +4 \\ \hline 8 \end{array}$$

entirely different addition facts are used when the direction of adding is reversed.

The system of checking subtraction is the same that it has been; that is, add the difference to the subtrahend to see whether their sum equals the minuend. Some texts suggest that this be done on the existing written example; others have the student rewrite it. In either case, this procedure gives the teacher an excellent opportunity to review students' understanding of the meaning of subtraction.

Problems. Students of earlier generations frequently expressed dislike for "word" problems. Possibly one reason for this was that sets of problems were relatively scarce in their arithmetic books, the emphasis being on long columns of exercises. The modern third-grader, however, has little opportunity to become complacent over the scarcity of problems. Hardly a page comes up that doesn't present problems of some sort. Additionally, most teachers are constantly looking for class activities that can serve as bases for real problems. One third-grade teacher in a combined elementary-high school builds numerous problems around sports activities with excellent results.

Teaching materials. The use of place-value pockets in carrying and borrowing has already been described. There is a definite place at this point in arithmetic for records, flash cards, and other drill materials. Now that all of the addition and subtraction facts have been introduced, it is vital that extensive practice be used.

Parts of several previously listed films and filmstrips could be used at this point. An interesting variation would be the Speed-O-Strip Series. Included are a set of filmstrips on addition combinations and another on subtraction combinations designed to help students develop facility in working with the basic facts in these processes. These filmstrips are available from the Society for Visual Education.

MULTIPLICATION AND DIVISION

When these operations are first encountered by third-graders they are usually dealt with as though they were new topics. Hence, we

usually begin with basic concepts; that is, that multiplication is a procedure for adding equal groups to form a new group, whereas division is a short method for performing equal subtractions. Many illustrations are used, possibly with some concrete materials, in order to establish desired concepts of multiplication and division.

Facts. Since the processes of multiplication and division are usually introduced in second grade, it is essential that some of the facts be used. Few programs, however, go beyond a few of the facts that are used illustratively. Hence, in third grade, nearly all the multiplication facts are yet to be learned.

One approach sometimes used in the teaching of multiplication in third grade begins with adding equal groups. From here, the students move to the concept that $4 + 4 + 4$, for example, may be thought of as 3 fours, or 12. This transition is made gradually, with frequent reference back to the addition process. As the class moves into multiplication specifically, they begin to use the $\times$ symbol and to write exercises, as

$$\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array}$$

In most programs, however, students continue to describe the operation as "3 fours" rather than "3 times 4."

Most of the texts begin to introduce the facts after a short review, and the system used varies extensively. One widely used technique is to teach facts by families, such as

$$\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array} \quad \text{and} \quad \begin{array}{r} 3 \\ \times 4 \\ \hline 12 \end{array}$$

It should be pointed out that, although these two exercises yield the same product, the operations are actually quite different. One deals with three groups of 4 each, while the other involves four groups of 3 each. Sometimes reference is made to the zero facts, such as

$$\begin{array}{r} 3 \\ \times 0 \\ \hline 0 \end{array} \quad \text{and} \quad \begin{array}{r} 0 \\ \times 3 \\ \hline 0 \end{array}$$

Since there is no way to make the facts meaningful, however, this procedure is of questionable value.

Some parents of third-graders wonder when their youngsters will get to the multiplication tables, a device of vivid, but probably not very fond, memory. The answer, as implied earlier, is that a student seldom actually sees the whole formidable array of facts. Rather, he studies them in "smaller doses" over a period of almost three years. Incidentally, there is no good explanation of the emphasis which earlier texts placed on the multiplication tables when the equally important addition, subtraction, and division facts were given far less emphasis.

Although there is little research to indicate the best way to introduce the multiplication facts, we commonly use the multiplier of 2 as a start. This actually means more to most students than would a multiplier of 1. From there, the list is expanded by families. Some programs limit the number of facts for third grade to about thirty; others go to sixty or more. Probably the deciding factor would be how many the class could effectively learn.

In some cases, division and multiplication are taught as closely associated operations with many mutual learnings. In other programs, the two operations are taught separately. (In earlier work, the exercises are shown as $4\overline{)12}$, with the symbol $\div$ being introduced later in third grade.) It seems reasonable that a student could learn that $\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$ ("there are 3 fours in 12") and that $\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$ with less effort when he is learning that

$$\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array} \quad \text{and} \quad \begin{array}{r} 3 \\ \times 4 \\ \hline 12 \end{array}$$

than he could later. So the teacher is faced with the problem of keeping these two operations close enough together to use mutual learnings, but far enough apart to minimize confusion.

In division, the remainder presents a new difficulty. Some third-grade texts introduce this idea and give students some simple, usually concrete, work with remainders.

Since for every grouping operation in multiplication there is an un-grouping operation in division, most texts present the same number of facts in these two processes. A key point is that third-graders are beginning the process of mastering these facts. Hence, many types of recurring experiences (drill) should be used.

Higher decade products and dividends. As students work at learning the basic facts, they continue to expand concepts. One such expansion

involves two-place numerals in the multiplicand and dividend positions. Here, incidentally, the zero demands attention, as in

$$\begin{array}{r} 10 \\ \times 4 \\ \hline 40 \end{array}$$

It should be explained that 4 zeros yield zero. Now that the zero is used as a part of a two-place number, however, we have a situation that can be verified concretely. This is not true of $\begin{array}{r} 0 \\ \times 4 \\ \hline \end{array}$.

Most of the programs that use two-place numerals, as just shown, select those that do not require carrying, since this introduces the concept without the complications. Such operations as $\begin{array}{r} 12 \\ \times 3 \\ \hline \end{array}$ are actually little more difficult than one-place numbers would be. In some cases, even three-place products are produced, such as

$$\begin{array}{r} 53 \\ \times 2 \\ \hline 106 \end{array}$$

Again, this involves no major new concept and is not, therefore, particularly difficult for most students. Correspondingly, several programs use exercises like $2\overline{)44}$, or even $2\overline{)248}$, since these involve no major difficulty beyond those found in simpler examples.

Checking results. Many adults may recall that, in their arithmetic books, there were whole pages of exercises with just three words, such as "Subtract and Check." Obviously, the authors rated the two processes as equally important.

There has been considerable change in our attitude toward checking operations. If a youngster knows that he must check his work anyhow, what incentive has he to get it right the first time?

Several problems have arisen as a result of stress on checking. One, for example, concerns the student who merely goes through the motions. If he subtracts $\begin{array}{r} 43 \\ -21 \\ \hline \end{array}$ and gets 19, he calmly scribbles $\begin{array}{r} 19 \\ +21 \\ \hline \end{array}$ and gets 43, since he knows that is what he should get. He may feel that he has lived up to the letter if not the spirit of the requirement. Further, isn't it true that great emphasis on checking each operation constantly confronts the student with the likelihood of error? Some teachers are now taking the attitude that the requirement to check endlessly can

actually become a sort of crutch. Therefore, we find that many programs show students how to check and require them to do a limited amount of checking. But it is far more important that a student get the correct result the first time than that he be adept at various techniques for locating errors. If he does not make errors, obviously he will not have to locate them.

Teaching materials. Since the first major emphasis on multiplication and division comes in third grade, there is considerable use for concrete materials. Most of these should be very simple, such as counters or blocks. Further, most teachers look for opportunities to use concrete materials found in school activities, such as books, cookies, milk cartons, and pieces of chalk.

The types of material mentioned earlier; that is, flash cards, records, the Arithmequiz, and other drill devices, are most helpful. Certain frames of previously listed filmstrips are useful. The filmstrip series called Speed-O-Strip, described earlier in this chapter, has a set on multiplication and one on division. Although these are designed primarily to help develop speed, they can also be used in drill to develop general facility with the multiplication and division facts.

FRACTIONS

Certain fraction concepts were introduced in grades one and two. These are usually retaught, with some expansion of the meaning of fractions, at third grade.

A popular approach is to begin fraction work concretely, with one-half as the fraction studied. With constant reference to the meaning involved, the teacher leads the class to work with other unit fractions, usually thirds, fourths, and possibly fifths. It is common practice to limit this work to unit (numerator of 1) fractions at third grade. Further attention is given to the very important point that if a pie is to be cut into fourths, it must be cut into four *equal* parts.

Some texts introduce the numerical symbols for fractions in third grade. Heretofore, the common practice has been to use the word forms "one-half" or "one-fourth." In presenting the numeral form, it is usually pointed out that $\frac{1}{3}$ of an apple means $\frac{\text{1 of the}}{\text{3 equal parts}}$ into which the apple is divided. This, of course is rapidly shortened to $\frac{1}{3}$.

Most third-grade programs continue to concentrate on fractional parts of a single object. A few of them, however, expand into the

somewhat more complex work involving fractional parts of groups. A few even introduce some elementary operations with fractions, such as $\frac{1}{2}$ of 12 or $\frac{1}{3}$ of 6. Obviously, these would be meaningless unless preceded by a study of fractional parts of groups.

Teachers constantly try to assure that students will not associate a particular fraction with a shape of an object. Thus, in showing one-half of a sheet of paper, students should be given an opportunity to see how many different ways they could fold or cut the sheet in order to produce halves.

Problems. Almost all the texts use problem situations to lead into the study of fractions. Some teachers object to the extensive use of foods (fruit, cakes, pies, candy bars) in such problems. One criterion of a good problem, however, is that the students find it interesting. What could be more interesting to the typical third-grader than food?

Actually, the problem approach is usually limited, at this phase of the study of fractions, to problem situations using familiar materials, with questions of a semiquantitative nature. The actual solution of problems through number manipulation is usually reserved until a later grade.

Teaching materials. Reference has already been made to such teaching materials as fraction boards, flannel boards with fraction cutouts, and simulated fruit divided into fractions.

Some of the most effective materials for this work are available in a normal classroom. Some activities might be (1) cutting a sandwich or slice of bread in half; (2) sawing a board into halves, thirds, or fourths; (3) cutting a string into halves, thirds, or fourths; (4) devising a variety of methods for cutting paper into designated fractional parts; (5) applying fractional terms to measuring cups; (6) coloring one-half of a square or circle. The classroom teacher could extend the list almost indefinitely.

A limited number of frames of certain filmstrips could be used at third-grade level. Some of these are (1) "Meaning of Fractions," by Young America Films; (2) "What is a Fraction?" by Filmstrip House; and (3) "Meaning of Fractions," by The Society for Visual Education.

MEASUREMENT

We expect third-graders to work with several kinds of measures. Further, they begin to develop facility, hence more accuracy, with the measures studied in earlier grades.

Money. Third-graders may work with all the coins up to, and including, the dollar. There is less emphasis on the concrete than was the case in grade two. Pictorial material, however, is used extensively.

Some of the activities with money are making change for a purchase, changing one coin into equivalent sums with other coins, carrying out the fundamental operations with money, and using the decimal point in indicating dollars and cents. Since the modern third-grader has some real-life experience with money, quite unusual a few years ago, the work with money has more air of reality.

Time. The time measures studied in second grade are retaught. The relationship between the minute and the hour is developed. Later, the students move into the study of the day, week, month, and year.

Probably no kinds of measures are more confusing to youngsters than are time measures. One factor is the lack of concreteness. Further, our time measures have certain characteristics that contribute to confusion; for example, all months are not the same length. Further, although third-graders do not realize this, our ninth month (September) has a prefix indicating 7; October has a prefix indicating 8; and November, though it is the eleventh month, has a prefix indicating 9. *Do you know the reason for this?* Other features of time measures frequently studied in third grade are reading the clock to a quarter-hour, along with A.M., P.M., noon, and midnight.

Distance. The distance measures for third grade usually include the inch, foot, and yard. Considerable attention is given to conversion from one unit to another. Also, students usually practice using the foot rule and yardstick in order to develop a concept of accuracy in measurement. There are, of course, numerous opportunities to use distance measures in the classroom and on the playground.

Volume. The list of units studied in this phase of measurement varies considerably among the texts. The half-pint is usually included, since this is the size of milk bottles or cartons served in many school lunchrooms. Further work is usually taken up using the pint, quart, and gallon. Much of this is based upon actual manipulation of these measures, usually in connection with a realistic problem.

Weight. The weight measures usually studied at this level are the ounce and the pound. The earlier concepts are retaught, and much of the study of weight is done concretely. The weight of the student is frequently used as a starting point, with the ounce being brought in

later. Adding weight quantities is a frequently used activity, with some form of kitchen scale serving as an important teaching aid.

Aids available. Probably no phase of arithmetic is more abundantly supplied with teaching aids than is measurement. It is almost automatic that our study of distance includes rulers and yardsticks; that our study of time includes clocks and calendars; that in our study of weight we use scales or balances; and that in our study of volume we use measuring cups and assorted containers.

There are some excellent filmstrips that could be used, in whole or in part, at this level:

The Society of Visual Education has several filmstrips, such as,

1. "Learning to Tell Time"
2. "Learning About and Using Pennies, Nickels, and Dimes"
3. "Using and Understanding Simple Measures"
4. "Using and Understanding the Calendar"

Filmstrip House has a set of four filmstrips in a series called Man and Measures.

Popular Science has six filmstrips in a set called Units of Measurement Series.

Young America Films has a very interesting series (six filmstrips) entitled, "History of Measures."

Several films dealing with measurement are well suited for lower elementary classes. For example, Coronet Films has the following:

1. *The Calendar: Days, Weeks, Months*
2. *Let's Measure: Inches, Feet and Yards*
3. *Let's Measure: Ounces, Pounds and Tons*
4. *Let's Measure: Pints, Quarts and Gallons*
5. *Making Change for a Dollar*
6. *What Time Is It?*

C. The Slow Learner

By third grade, the wide range of student ability is beginning to appear, and teachers, early in the school year, can tell which students will be likely to require extra help. Certainly, it is to the student's advantage if he can get such assistance early in the school term. Some of the special techniques for use with the slow learner are more emphasis on the concrete, more drill than is

necessary for average students, and special effort to see that some success is experienced.

In third grade, students are expected to achieve mastery of certain basic facts in the four fundamental operations. Failure to do so means more difficulty in the future. It is, therefore, important that slower-learning students concentrate on the basic facts in the four operations.

It is vital that the teacher give special attention to spotting difficulties through diagnostic testing and especially through careful observation of the student at work. Listening to a student as he works aloud is an especially effective technique. To illustrate, a third-grade girl was having difficulty with subtraction. By listening to this student as she talked through a few exercises, the teacher detected that the child had never learned to borrow but merely subtracted the smaller from the larger number. In an exercise like $\begin{array}{r}41\\-16\\\hline\end{array}$ she would subtract 1 from 6, then 1 from 4, getting a difference of 35. Although the result was wrong, this procedure made the mysterious process of borrowing entirely unnecessary. Another important practice is for the teacher, in checking a student's paper, to go beyond marking results right or wrong and to look for patterns of errors on tests and seat-work papers.

Obviously, diagnostic and remedial techniques require individual procedures. Mass diagnosis is about as valuable in arithmetic as it would be in prescribing medicine or fitting spectacles.

Some of the textbooks, either regular or teachers' editions, give special help to the teacher in her work with slow learners. One procedure is suggesting special activities for slower learners at the end of each section. Another is incorporating diagnostic tests at regular intervals in the text, with references to supplementary drill exercises that would help correct certain deficiencies.

One other help would be having available texts of varying degrees of difficulty, probably including some books or workbooks for second grade. Frequently, these can save the teacher the time and effort necessary to devise exercises for a slower student.

D. The Rapid Learner

It usually becomes apparent by third grade that certain students are especially adept at learning arithmetic. Many teachers find such students to be serious problems, since they finish assignments easily and thus have time on their hands.

Certain texts make provision, at least in part, for these students by incorporating such features as "How Far Can You Go in Arithmetic?"[1] Others have a special set of exercises at the end of each topic which contains challenging questions for the rapid learner.

Frequently, teachers give rapid learners report assignments that are of interest to the class as a whole. In one particular third grade, a student gave a report on the history of Roman numerals that was enlightening to his class, his teacher, and especially to a group of college students who were observing his class.

We have referred to some valuable helps for the teacher in challenging the rapid learner. One that is widely used is H. F. Spitzer's *Practical Classroom Procedures for Enriching Arithmetic*, published by the Webster Publishing Company. Many of the activities described in this book are useful at third-grade level.

Another valuable aid is the set of enrichment booklets published by Harper and Row, Publishers. These are relatively inexpensive, well illustrated, and probably interesting to most third-graders. Some titles from this series are

1. "Number Stories"
2. "Riddles and Puzzles"
3. "Crossnumber Puzzles"
4. "The Story of Counting"
5. "Tricks with Picks"
6. "Fun with Words"
7. "Nimble Numbers"

A word of precaution should be given concerning the rapid learner. His side excursions in arithmetic should be in addition to, rather than in lieu of, the work being done by the other students. He, too, needs a basis in fundamentals; the big difference is that he can acquire this basis more easily than can some of his classmates. Consequently, the supplementary work should come after the rapid learner has completed his study of the fundamentals of arithmetic.

Something to Think About

1. Suppose that, after you have explained to your third-grade class that multiplication is a special type of addition, a student asks why he should bother to learn it, since he already knows how to add. How would you answer?

[1] The Silver Burdett series, of which Robert Lee Morton is senior author.

2. This question is asked in class: "If the zero is so important, how did the Romans get along without it?" How would you respond?

3. Make a list of patterns or principles that a third-grade class might observe in examining a hundred board.

4. Who was the best mathematics teacher you ever knew? What qualities did he or she possess that made him outstanding?

5. What would be your reaction if you learned that you were teaching Mary one method in borrowing while her mother was teaching her another method?

6. Devise a system for demonstrating concretely the difference between bridging and carrying.

7. If a parent asked you just what is so objectionable about finger-counting, how would you answer?

8. If you can locate an elementary arithmetic text that dates back at least half a century, contrast the presentation of carrying and borrowing in it with the comparable presentations in a modern text.

Selected References

Banks, J. Houston. *Learning and Teaching Arithmetic.* Boston: Allyn and Bacon, Inc., 1959.
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McSwain, E. T., and Ralph J. Cooke. *Understanding and Teaching Arithmetic.* New York: Holt, Rinehart and Winston, Inc. 1958.
A thorough treatment, with emphasis on understandings.

Myers, Sheldon S. *Published Evaluation Materials in Mathematics.* Washington, D.C.: The National Council of Teachers of Mathematics, 1961.
An annotated bibliography of commercially available tests in mathematics, including arithmetic.

Harper and Row, Publishers, has a set of eight enrichment booklets specifically designed for use in third grade.

Spitzer, Herbert F. *The Teaching of Arithmetic.* Boston: Houghton Mifflin Company, 1961.
The chapter on differentiated instruction is especially well done.

———. *Practical Classroom Procedures for Enriching Arithmetic.* St. Louis: Webster Publishing Co., 1956.
Has excellent enrichment activities suitable for third grade.

Humanistic Concepts for The Teacher

CHAPTER SIX

In earlier times, many concepts of number and number systems were accepted as intuitively obvious. When many of these *obvious* facts were shown to be false, mathematicians began to question intuition as a basis for knowledge of number.

Going to the opposite extreme, the formalists attempted to make mathematics a game played according to accepted rules. The objects used in the game were the number symbols. Thus, the number 1 was the symbol "1," and so on. Somehow this does not seem to fit our understanding of number, since we actually do attach a meaning to number symbols outside the symbols themselves.

The logical approach seems to offer better opportunities for defining and developing an understanding of number operations. Within the framework of a theory of sets, number and number operations may be defined in a way which at once satisfies our intuitive concepts of number and our desire to make number a logically derived concept.

This chapter presents a brief view of how computation has developed followed by a logical development of the rules for number operations. Also, a development of the integers, rationals, and real numbers from the natural number is made.

Topics presented are:

A. The Development of Computation

Is computation something we do with numbers, or is it something we do with symbols for numbers? Should we always carefully differentiate between numbers and numerals when doing computation? The answer to these questions depends on our beliefs about numbers. If we tend toward the formalists' viewpoint, then numbers are the symbols we use. If, on the other hand, we tend toward the logicists' viewpoint, then numbers have an existence apart from the symbols used to designate them. Elementary arithmetic is ordinarily concerned with finite numbers and operations on these numbers. Much of what goes by the name "computation" is concerned with manipulating our base-10 system of numerals to find standard names for numbers named in some kind of sentence or expression.

EARLY ARITHMETICAL TERMS

In ancient times, *arithmetic* was the term used by Greeks and Romans to refer to number theory. Computation, or the art of calculating, was designated by the term *logistic.* Smith[1] noted that these two branches of number study were considered as separate subjects until the sixteenth century, when the name *arithmetic* came to be applied to both.

Early arithmetic, based on the Hindu-Arabic base-10 notation, was called *algorism.* Other terms that have been used for computation are *reckoning, ciphering,* and *practical arithmetic.* The numbers 1 through 10 were at first referred to as the *digits.* Most literature now refers to the numbers 1 through 9 as the digits. Very careful writers point out that the symbols 1 . . . , 9 are the digits. At one time, unity was not considered a number since numbers were quantities or pluralities made up of unities.

From very ancient times, there has been a distinction between odd and even numbers. Prime numbers, numbers divisible only by themselves and 1, were also defined in ancient times. The Greeks classified composite (not prime) numbers as deficient, perfect, or abundant. These names were given to numbers which were greater than, equal to, or less than, the sum of their exact divisors, including 1 but not themselves. Six is a perfect number since $1 + 2 + 3 = 6$. *How would you classify* 12? 15?

[1] David Eugene Smith, *History of Mathematics,* Vol. II (New York: Dover Publications, Inc., 1953), p. 7.

The ancients made a distinction between cardinal and ordinal numbers. Most of the modern developments in number belong to higher mathematics and come under the general term *number theory*. Most of these developments are of little concern to elementary arithmetic.

THE FUNDAMENTAL OPERATIONS

We commonly speak of the four fundamental operations. These are, of course, addition, subtraction, multiplication, and division. The operations were what the Greeks and Romans called *logistic*. In earlier times, some writers recognized eight or nine "fundamental operations." Other writers have argued that there should be only one fundamental operation, that of counting or addition.

Addition and subtraction. Early terms used for addition included *aggregation*, *collection*, and *summation*. These have a surprisingly modern sound since they are used in describing sets. The word "sum" has come to be used for the result obtained in addition, although many other terms have been used.

In Chapter 2, we presented an intuitive development of rules guiding number operations. Later in this chapter, a logical development of the rules for the operations on natural numbers will be shown. These rules will be a logical consequence of our definitions regarding sets and natural numbers. In the past, number operations usually referred to the practical arrangements for using a particular system of numeration. We now attempt to distinguish between a "number operation" and the process of finding a standard name in our set of numerals.

The so-called operation of addition has changed little since the use of Hindu-Arabic numerals became widespread. Generally, the Hindus wrote out the names of the number places and found the sums in each place. The sums by places were then combined to give the final sum. The carrying concept is an old one and dates from the time of the line abacus.

Early terms used in connection with subtraction included *extract*, *diminish*, and *rebate*. The terms *difference* and *remainder* are now commonly used in textbooks. Unlike the process of addition, the process of subtraction has not been standardized in our system of numeration.

Several plans for subtraction have been widely used. The complementary plan for subtraction was taught in this country in the last century and was known much earlier. The *complement* of a number is

the result obtained when the number is subtracted from the power of 10 just larger than the number. The complement of 86 is 14, since $100 - 86 = 14$. That of 6 is 4, since $10 - 6 = 4$. This process may be illustrated as follows:

513	$10 - 6 = 4$	$4 + 3 = 7$
-246	$10 - 5 = 5$	$5 + 1 = 6$
267		$5 - 3 = 2$

The equal-additions method has been known and used for several centuries. It is sometimes called the *borrowing* and *repaying* method. It may be illustrated as follows:

513	6 from 13 is 7
-246	5 from 11 is 6
267	3 from 5 is 2

The decomposition or simple borrowing method is also several centuries old. It consists of regrouping where necessary. This process involves the following steps:

513	6 from 13 is 7
-246	4 from 10 is 6
267	2 from 4 is 2

Notice that regrouping was done in the tens' and hundreds' places. One ten was regrouped to 10 ones, and 1 hundred was regrouped to 10 tens.

The additive method of subtraction was proposed several centuries ago, but did not find much use until the last century. The steps in this process are as follows:

513	6 and 7 are 13
-246	5 and 6 are 11
267	3 and 2 are 5

Multiplication and division. Addition of whole numbers is a way of obtaining certain results without the longer process of counting. Multiplication was recognized very early as a short cut for addition. Early definitions called multiplication "an operation on two numbers to get a third number such that the third number contained the first number as many times as there are units in the second number." Such definitions did not reveal very much about the process of multiplication.

The terms *multiplicand*, *multiplier*, and *product* had their origin in early Latin words. Little is known of the process of multiplication in ancient times. Ancient peoples probably made use of the duplation plan for multiplying. This is illustrated as follows in finding the product of 14 and 27:

27	54	108	216	432
1	2	4	8	16

Then 432 − 54 = 378, the product of 14 and 27.

Russian peasants in very recent times have been known to use duplation and mediation. This plan may be called *doubling and halving*. It is illustrated by finding the product of 27 and 35. Halve 27, discarding remainders, and double 35.

27	35	35
13	70	70
6	140	
3	280	280
1	560	560
		945

To find the product, add the terms in the "double" column that stand opposite odd numbers in the "halve" column. This plan works on the principle that if you halve one factor and double the other, the product remains the same. Therefore, 27 × 35 would be the same as 1 × 560 except that we lose some each time we drop a remainder in the halving process. Since we lose some each time we have an odd number in the "halve" column, we must add to 1 × 560 the amount lost, which is the amount opposite each odd number in the "halve" column.

Another early plan for multiplying was the cell or grating method. The numbers to be multiplied are written, one horizontal and one vertical, about a rectangle ruled into cells. Diagonal lines are drawn through the cells. For example, 78 × 25 (see Fig. 6.1). Each product of digits is written in a cell, with the tens digit in the upper half of the cell and the units digit in the lower half. Then addition is done along the diagonal strips, carrying if necessary over to the next diagonal strip to the left.

Many other plans for multiplying have been used. Early Arabs knew and used various algebraic relations to aid multiplication of whole numbers. The multiplication table has been found on Babylonian cylinders in columns. The square form was used in early Latin writings.

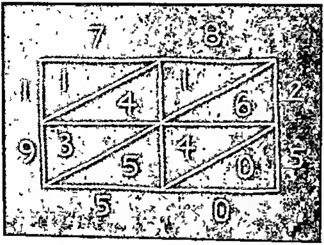

Figure 6.1

Division, like multiplication, has no definitions which can be understood by beginners. The concept must grow out of concrete examples. Early definitions, such as one which stated that division showed how often the lesser number is contained in the bigger, contribute little to the understanding of the operation.

One of the older methods of division was based on the process of duplation and mediation. To divide 42 by 16, proceed as follows:

1	16
2	32
$\frac{1}{2}$	8
$\frac{1}{4}$	4
$\frac{1}{8}$	2

Set down 1 opposite 16, then $2 \times 16 = 32$; $\frac{1}{2}$ of $16 = 8$; $\frac{1}{4}$ of $16 = 4$; and $\frac{1}{8}$ of $16 = 2$. The numbers in the right-hand column whose sum is 42 are 32, 8, and 2. Therefore, the quotient is $2 + \frac{1}{2} + \frac{1}{8}$.

Another method used in the Middle Ages made use of the factors of the divisor. The object was to obtain one-digit divisors which could be handled. For example, to divide 288 by 12 would be the same as $288 \div 4 \div 3$.

A galley or scratch method was used in Europe for several centuries. It is thought to have originated with the Hindus. The steps in this method are rather easy. A bit more mental arithmetic is involved than in our traditional long division method.

Our present method of long division developed gradually. Its origin is difficult to fix. Various ways of placing the quotient have been used. The present practice of placing the quotient above the dividend seems to work well and allows easy location of the decimal point.

Other number operations. Early writers included such concepts as numeration, notation, duplation, mediation, progression, and extracting of roots among number operations. Other than what we consider the four fundamental operations, extracting of roots and raising to powers may now be considered rather basic operations with numbers.

The Greeks had a rather crude method of finding approximate square roots. The Arabs also had a method, similar to that used by the Greeks. A galley method has been found in early arithmetic books. As in division, this is a rather complicated process involving much mental "arithmetic." Other procedures were tried, and gradually our present square root algorithm came to be used. The present procedure for extracting roots is rather complicated and difficult to explain to children who have had no algebra. For this reason, an explanation of the square root rule is usually postponed until pupils have had some experience with algebra.

Successive multiplications by the same number led to the idea of raising numbers to powers. We write powers as follows:

$$\begin{aligned}
4^1 &= 4 \\
4^2 &= 4 \times 4 &&= 16 \\
4^3 &= 4 \times 4 \times 4 &&= 64 \\
4^4 &= 4 \times 4 \times 4 \times 4 &&= 256
\end{aligned}$$

This is a powerful notation for writing or symbolizing large numbers. Early attempts to write large numbers are known. Archimedes, in *The Sand Reckoner*, presents a way of conceiving and writing large numbers though hampered by the lack of zero in the Greek alphabetic way of writing numerals. He used his system to show a number larger than the number of grains of sand needed to fill the universe (his definition of universe).

A googol has been defined to be 10 raised to the hundredth power. That is 10^{100}. This is a number denoted by a 1 followed by 100 zeros. Can you conceive of the number denoted by 10^{googol}? This is 10 raised to the googol power. This number would be denoted by a 1 followed by such an array of zeros that we cannot conceive of the quantity at all.

Raising to powers and extracting roots are inverse operations, as are addition and subtraction or multiplication and division. For example,

$$5^2 = 25 \quad \sqrt{25} = 5 \quad \text{and} \quad 4^3 = 64 \quad \sqrt[3]{64} = 4$$

The notations for powers and roots find many useful applications in algebra and advanced number study.

Checking operations. Checking by using inverse operations had a very early origin. Some definite references to inverse checking may be found in the Middle Ages. The inverse operation method of checking was long, and simpler methods were developed in early times.

The method of casting out nines is perhaps the best known. Early Arab writers recognized and used this method. Early American schools used casting out nines, but it disappeared from textbooks for quite a while. More recently, it has appeared again in arithmetic books.

Casting out nines as a method of checking number operations is based on some properties of the number nine. A number which is a multiple of nine has digits whose sum is nine or a multiple of nine. For example,

18	$1 + 8 = 9$	
36	$3 + 6 = 9$	
99	$9 + 9 = 18$	$1 + 8 = 9$
387	$3 + 8 + 7 = 18$	$1 + 8 = 9$
8478	$8 + 4 + 7 + 8 = 27$	$2 + 7 = 9$

Likewise, the remainder, when a number is divided by 9, is equal to the remainder when the sum of the digits of the number is divided by 9. When 77 is divided by 9, there is a remainder of 5. If we add the digits in the number 77 to get 14 and divide this by 9, the remainder is 5. How can this be used to check number operations? Below is an addition example checked by casting out nines.

247	$2 + 4 + 7 = 13$	$1 + 3 = 4$
+381	$3 + 8 + 1 = 12$	$1 + 2 = +3$
628	$6 + 2 + 8 = 16$	7
	$1 + 6 = 7$	

Digits may be successively added until one of the numbers 1 to 9 is secured. The sum of the remainders in the two addends equals the remainder in the sum.

In a similar way, casting out nines may be used to check other number operations. The general rule may be stated as follows: Find the remainders in all the original numbers when nines are cast out. Do the required computation with these remainders. The remainder of

this result when nines are cast out should equal the remainder found in the result of the original computation when nines are cast out.

An example in multiplication is as follows:

479	$4 + 7 + 9 = 20$	$2 + 0 = 2$
$\times 38$	$3 + 8 = 11$	$1 + 1 = \times 2$
3832		4
1437		
18202	$1 + 8 + 2 + 0 + 2 = 13$	$1 + 3 = 4$

Division may be checked as follows:

$$\begin{array}{r} 25 \\ 34\overline{)876} \\ \underline{68} \\ 196 \\ \underline{170} \\ 26 \end{array}$$

Note that divisor times quotient plus remainder equals dividend.

25	$2 + 5 = 7$			
34	$3 + 4 = 7$	$7 \times 7 = 49$	$4 + 9 = 13$	$1 + 3 = 4$
26	$2 + 6 = 8$	$\longrightarrow 8 + 4 = 12$	$1 + 2 = 3$	
876	$8 + 7 + 6 = 21$	$2 + 1 = 3$		

One difficulty with casting out nines as a way of checking a number operation is that, if two digits are transposed to give an incorrect answer, the result will check out correct. Is this method something characteristic of natural numbers or is it something characteristic of the numerals used in the Hindu-Arabic decimal system of numeration? Do you find the sum of the digits in a number or in a numeral? Can you write numbers using bases other than 10 to clarify this method? Have we used the word "number" in this section when we really meant "numeral?"

AIDS TO COMPUTATION

Many different devices to aid in computing with numbers have been used since ancient times. These range all the way from using the fingers and various forms of the abacus to Napier's rods, slide rules, and modern mechanical calculating machines. More recently still, the electronic computer has been designed to do literally thousands of calculations in a matter of minutes.

Finger reckoning. Most early peoples had a system of number notation in which the fingers were used.[2] From this notation came finger computation or "finger reckoning." From simple counting and adding, a way of doing multiplication of small numbers was worked out. Such numbers as 12 and 14 could be multiplied. For example, raise two fingers on one hand and four fingers on the other. Then, to 100, add ten times the sum of the number of fingers raised, and add the product of the number of fingers raised on the hands. This gives:

$$100 + 10(2 + 4) + (2 \times 4) = 168$$

This may be seen from the fact that:

$$(10 + x)(10 + y) = 100 + 10(x + y) + xy$$

Until very recently, many European peasants used finger reckoning and so could do simple multiplication, knowing only the multiplication facts to 5×5.

The abacus. It was thought that the earliest abacus was a table or stone slab covered with dust or sand. Lines and figures were drawn in the dust with a stick and erased with the finger. Later, a table with ruled lines and some kind of markers or counters were used to indicate numbers and to do number operations. The table top with grooves or rods on which markers or counters were placed may still be found in some parts of the world. The frame with wires or rods on which movable beads are placed may be found in many arithmetic classrooms in our country today.

The rod abacus with movable beads has taken a number of forms in different countries. In each case, beads are moved to a certain position to denote a number. The Chinese *suan-pan* took the form illustrated in Fig. 6.2. The rods from right to left represent units', tens', hundreds',

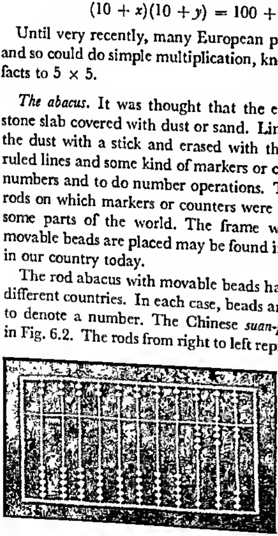

Figure 6.2

[2] *Ibid.*, pp. 196–202.

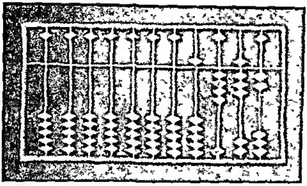

Figure 6.3

thousands' places, and so on. Beads below the horizontal bar are worth one each of the unit represented by their rods. Those beads above the horizontal bar are worth five each of the unit represented by their rods. The number 5482 is noted in Fig. 6.2.

The Japanese *soroban* differs slightly from the *suan-pan*. Again, the number 5482 is illustrated on the abacus (Fig. 6.3). Addition is carried out by simply setting up the first number on the abacus, and then adding the digit found in each place of the second number. When ten counters or beads are reached at each place, they are pushed back and one counter is moved into position at the next place on the abacus.

A simple sort of abacus is found in many classrooms today. Ten beads are found on each rod. The usual place value of ones, tens, hundreds, and so on, are used. Beads are pulled down or toward the child to denote a number. The number 163 is illustrated in Fig. 6.4. The abacus has been found very useful in developing certain number concepts.

Other devices. Napier's rods, developed in the seventeenth century, were an improvement over the older counter or line abacus method of computation. The system of rods made use of the grating method of multiplication and still was a rather crude method of computing. The

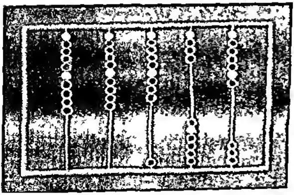

Figure 6.4

slide rule was also first used in the seventeenth century. Based on logarithms, the distances between numbers were proportional to the logarithms of the numbers. Therefore, by adding or subtracting distances along the scales, it was possible to multiply or divide numbers. Various refinements were made, and the slide rule became very popular.

Calculating machines seem to have had their start in the seventeenth century with the construction of several machines by Pascal. These early machines used a disk which after being turned nine steps engaged a second disk and turned it one step as the first disk turned the tenth step. This process was continued for as many disks as needed for large numbers. Pascal's wheel or disk machine was built to help his father who was a superintendent of taxes in Paris.

Early in the nineteenth century, Babbage, in England, got an idea for computing tables of data with machines. He completed a small model, but support was withdrawn for construction of a larger model. Babbage continued to study the idea of "carrying" in addition and produced a mechanical method of carrying. He conceived the idea of an analytical engine which would operate as a universal computer and which contained many ideas or features found in modern computers. Most of his drawings contained parts and specifications far too accurate to be manufactured by the machine tools of his day. Some of the early ideas on calculating machines still may be found in electric calculating machines used in offices.

Electronic Computers. Despite popular articles referring to the electric computer, as a giant brain, this simply is not so. Such machines are designed to perform certain predetermined operations on data which are fed into the machine. The term *electronic* is used because the major components of such a machine are electronic circuits. These are devices making use of vacuum tubes or transistors to control or modify the flow of electric current. Some mechanical devices are also used in such machines. These may consist of devices to handle punched cards or tape or revolving drums on which data may be stored.

Computers can perform a number of "built-in" operations, such as addition, subtraction, multiplication, or division. They perform these operations in any sequence and quantity on data fed to them in accordance with instructions previously fed to them by people. A problem is solved by computers only when a human being has first thought out a way to solve the problem, instructed the machine in how to solve the problem, and then fed the data for the problem to the

machine. This is called *programing*. Programing consists of breaking down a complex operation into a sequence of simple operations and then writing up this sequence into a set of instructions for the machine. These simple operations may consist of adding, multiplying, comparing, storing, taking from storage, and writing. When the complete sequence of simple operations has been performed, then the complex operation is accomplished.

This analysis of a problem into a sequence of simpler steps or operations is a necessary part of programing any computer. The translating of this sequence of steps into instructions for the computer is a job unique for each kind of computer. Such machines do not make it possible for people to forget number and number operations. To the contrary, their operation requires a very high degree of understanding of number and number operations. The high speed with which such machines operate makes it possible to process large quantities of data that simply could not be handled before.

MODERN CONCEPTS OF COMPUTATION

Modern mathematics has focused attention on the "structure" of number systems. This includes the basic definitions and operations necessary to a number system. Emphasis has been placed on "understanding" these basic definitions and operations. In the past, emphasis was placed on the processes by which operations on numbers could be accomplished.

Perhaps the most important idea in the modern approach to arithmetic is that mathematics is a form of language with its own symbols. Some symbols name the entities used in arithmetic, such as the numerals which stand for numbers. Other symbols, such as "$=$," "$<$," and "$+$," stand for relations and operations on numbers. Now, an understanding of the referents of the symbols used in mathematics is sought. In the past, stress was placed on manipulating the symbols to find some standard form.

A numeral expression like "$4 + 3$" names a number. When we say that "$4 + 3 = 7$," we are merely demonstrating another name for the number. Now, we also call "$4 + 3 = 7$" a basic addition fact. We make use of the basic addition facts to add larger numbers. The expression "$246 + 178$" names a number. The steps by which we find that "424" is the standard name for this number is called *addition*. It should be noted, however, that these steps are peculiar to our own set of

symbols. If these two numbers were expressed in Roman numerals, the process by which we found the standard name in Roman numerals would differ considerably from our usual process. Therefore, it should be realized that most of what we usually call *number operations* are really processes peculiar to our own system of numeration.

A real understanding of number, number operations, and processes by which standard forms of numerals are found makes up what we usually call *arithmetic*. Some writers even refer to the manner by which we add or multiply two numbers as "processing numerals" rather than adding or multiplying numbers. Swain[3] divides arithmetic into an object level and a symbolic level. He notes that some of our rules in arithmetic operate at the object or number level, and others operate at the symbolic or numeral level. We should know at all times the level on which we are working. Confusing these levels leads to some of the puzzles encountered in mathematics. At the symbolic level,

which is smaller?

35 or 7

Obviously, the "35" is smaller than the "7." At the number level, however, 7 is smaller than 35. Other examples involving various symbols and kinds of numbers may be found in references.

Computation, then, is primarily concerned with manipulating numerals so that a standard name may be found for an expression or statement.

B. Operations on Cardinal Numbers

In any consideration of sets of objects, a very natural question arises. If the set A and the set B are combined, what number may we associate with the combined set? The process of finding the number of the set formed by combining sets A and B and other sets related to sets A and B will be described.

ADDITION OF CARDINAL NUMBERS

Consider the sets A and B which have no elements in common. They are disjoint sets. Let a and b be their respective cardinal numbers.

[3] Robert L. Swain, "Modern Mathematics and School Arithmetic," *Instruction in Arithmetic*, Twenty-Fifth Yearbook (Washington, D.C.: National Council of Teachers of Mathematics, 1960), pp. 270–95.

Form the union of sets A and B, $A \cup B$. The cardinal number of $A \cup B$ is defined to be the sum of a and b. This is an operation on a and b called *addition*. The symbol "+" will be used to denote addition.

Suppose set A has the number 2, and set B the number 5. Then the union of sets A and B would have the number $2 + 5$. Now, the cardinal number $2 + 5$ must be somewhere in the sequence:

$$0, 1, 2, 3, 4, \ldots, 9, 10, 11, \ldots$$

By the process of counting the elements of the set $A \cup B$, we would find the number of the set to be 7. Therefore, we know that $2 + 5 = 7$. The symbol "=" is used here to mean that $2 + 5$ and 7 are symbols or names for the same number. A symbol for a cardinal number is said to be in *standard* form when it occurs among our ordered list of cardinal numbers.

Before proceeding with our development of number operations and laws, some conventions for making statements about numbers should be made.

STATEMENTS AND VARIABLES

In discussing sets and numbers, we describe and state facts in our language just as when other topics are discussed. Statements about sets and numbers may be written, however, by using special symbols that have been defined. So far, we have defined sets, cardinal numbers, and the symbols $=$, $\neq$, $>$, $\ngtr$, $<$, $\nless$, $\in$, $\notin$, $\cup$, $\cap$, as they relate to either sets or cardinal numbers.

Some examples of statements are:

$$5 > 3$$
$$a \in \{a, b, c, d\}$$
$$4 < 135$$
$$6 + 2 = 8$$

Statements may be either true or false. We assume that every mathematical statement is either true or false, but not both. This is based on the law of logic, called the law of *contradiction* or *excluded middle*. An important part of mathematics involves "proofs" of the truth or falsity of mathematical statements.

Frequently, we are concerned with statement-forms. We may state that $x < 3$. Is this statement true? We say that the answer depends on x. If we knew what number to put in the place of x, then we could

say whether the statement is true or false. In such statement-forms, x is called a *variable*. Suppose we allow x to be any cardinal number; then the set of cardinals is called the *domain* of x or simply the *universal* set. The domain of x is sometimes called the *replacement set*; that is, the set which may be used to replace x in a given statement-form. Now, all x's which make the statement true are called the *solution set.* Consider the statement-form $x < 3$. Let the domain of x be the cardinal numbers. We then have the following statements:

$$0 < 3, \quad 1 < 3, \quad 2 < 3, \quad 3 < 3, \quad 4 < 3, \ldots$$

Not all of these statements are true. The solution set for the statement-form is $\{0, 1, 2\}$.

Consider the statement-form $x + 3 = 8$. Let the domain of x be the set of cardinals $\{0, 1, 2, \ldots, 8, 9\}$. The solution set obviously is $\{5\}$. No other numbers in the domain of x make a true statement when substituted for x.

LAWS OF ADDITION

We may now state and prove two basic laws about the addition of cardinal numbers. The proof of these laws comes from our definitions of sets and numbers. The commutative law of addition is

$$x + y = y + x$$

where x and y are variables whose domains are the cardinal numbers. This simply means that the order in which two cardinal numbers are taken does not affect the sum. If this statement is false, then there exist cardinal numbers a and b such that $a + b \neq b + a$. Let A be a set with number a and B be a set with number b, then $a + b$ is the number of $A \cup B$, and $b + a$ is the number of $B \cup A$. $A \cup B$ and $B \cup A$, however, are the same set according to our definition of the union of sets. Therefore, $a + b$ and $b + a$ are numbers of the same set and, therefore, must be the same number. This contradicts our assumption that there exist cardinals a and b such that $a + b \neq b + a$. Therefore, we can say that there do not exist cardinals a and b such that $a + b \neq b + a$. Our law has been demonstrated or "proved."

The associative law for addition is

$$(x + y) + z = x + (y + z)$$

where x, y, and z are variables whose domains are the cardinal numbers. Consider the sets A, B, and C, with no elements in common and having

numbers a, b, and c respectively. Now note that $a + b$ is the number of $A \cup B$, and $(a + b) + c$ is the number of $(A \cup B) \cup C$. Also, $b + c$ is the number of $B \cup C$, and $a + (b + c)$ is the number of $A \cup (B \cup C)$. Our definition of union of sets means that $A \cup B$ is the set of all elements either in A or B, and when this set is combined with C, we have all the elements in A, B, and C. This is the same set as $A \cup (B \cup C)$. Therefore, the sets $(A \cup B) \cup C$ and $A \cup (B \cup C)$ are the same set. Since $(a + b) + c$ and $a + (b + c)$ are numbers of the same set they must be the same number.

LAWS OF MULTIPLICATION

Multiplication of cardinal numbers may be thought of as repeated addition. For example, 4 threes are $3 + 3 + 3 + 3$. The symbol for multiplication is "$\times$" or "$\cdot$" and the result is called the *product*. We write the preceding example "4×3" or $4 \cdot 3$. We will define multiplication in terms of sets. First we need to define the *product set*. The product set of A and B, written "$A \times B$," is the set of all ordered pairs such that the first element in each ordered pair belongs to A, and the second element in each ordered pair belongs to B. Consider the sets $A = \{1, 2, 3\}$ and $B = \{u, v\}$, then the product set $A \times B$ is the set of ordered pairs $\{1, u\}$, $\{1, v\}$, $\{2, u\}$, $\{2, v\}$, $\{3, u\}$, $\{3, v\}$. *Is the set $\{u, 1\}$ a member of the product set $A \times B$?*

Now let a and b be the numbers of the sets A and B respectively. Then the *product* of a and b, written $a \cdot b$, will be the cardinal number of the product set $A \times B$. In the foregoing example, the number a is 3, and the number b is 2. The number of $A \times B$ is 6. *Why?* Therefore, according to our definition $3 \cdot 2 = 6$. *How does this compare with our intuitive development of multiplication in Chapter 2?*

We may now state and prove two basic laws of multiplication. The commutative law is

$$x \cdot y = y \cdot x$$

where x and y are variables whose domains are the cardinal numbers. Let A and B be disjoint sets with numbers a and b. *Form the product sets $A \times B$ and $B \times A$.* These sets are not equal, but they are equivalent because their elements may be put into a one-to-one correspondence. *How?* They, therefore, have the same cardinal number. Now, note that $a \cdot b$ is the number of $A \times B$ and $b \cdot a$ is the number of $B \times A$, but these numbers must be the same. Therefore, $a \cdot b = b \cdot a$ for any cardinal numbers a and b.

The associative law for multiplication is

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

where x, y, and z are variables whose domains are the cardinal numbers. Let a, b, and c be the numbers of sets A, B, and C respectively. Form the product sets $A \times B$ and $B \times C$. Consider the sets $(A \times B) \times C$ and $A \times (B \times C)$. Each member of $A \times B$, say (r, s), is associated with each member of C, say t, to form an ordered pair $((r, s), t)$. Likewise, each member of A, say r, is associated with each member of $B \times C$, say (s, t), to give an ordered pair $(r, (s, t))$. We may pair $((r, s), t)$ and $(r, (s, t))$ to give a one-to-one correspondence between $(A \times B) \times C$ and $A \times (B \times C)$. Therefore, these product sets have the same number, and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, which demonstrates our law.

A third law relates multiplication to addition.

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

where x, y, and z are variables whose domains are the cardinal numbers. Let A, B, and C be disjoint sets with cardinal numbers a, b, and c Then $b + c$ is the number of $B \cup C$, and $a \cdot (b + c)$ is the number of the product set $A \times (B \cup C)$. Since B and C are disjoint sets, this product set consists of all ordered pairs which have a first element taken from A and a second element taken from B or C. We get the same product set, however, if we form the union of the product sets $A \times B$ and $A \times C$. The number of this union is $(a \cdot b) + (a \cdot c)$; therefore, we have shown that $a \cdot (b + c)$ is the same number as $(a \cdot b) + (a \cdot c)$, since both are numbers of the same set.

SPECIAL ELEMENTS AND CONCEPTS

There are several special elements among the set of cardinal numbers. Consider the number 0. Notice that $0 + x = x$ and $0 \cdot x = 0$, where x is any cardinal number. Let A be any set with number a. Now $0 + a$ is the number of the union of the null set and A. The union of the null set and A is A, whose number is a. Therefore, $0 + a = a$. Now, $0 \cdot a$ is the number of the product set formed by the null set and A. Since the null set has no elements, no ordered pairs can be formed and the product set is the null set whose number is 0. Thus, $0 \cdot a = 0$. We have demonstrated that $0 + a = a$ and $0 \cdot a = 0$ for any cardinal number a; therefore, our generalized statements about 0 are true.

Now, consider the number 1. Notice that $1 \cdot x = x$ where x is any cardinal number. Let A be any set with number a. This means that the elements of A can be put into a one-to-one correspondence with the set of cardinal numbers $\{1, 2, \ldots, a\}$. The number 1 is obviously equivalent to the set $\{1\}$. *Form the product set* $\{1\} \times \{1, 2, \ldots, a\}$. This produces the set of ordered pairs $(1, 1)$, $(1, 2)$, ..., $(1, a)$. Obviously this set can be put into a one-to-one correspondence with the set $\{1, 2, \ldots, a\}$ and the set A. Thus the number of the product set is a. Therefore, $1 \cdot a = a$, and our generalized statement has been established. It should be noted that no numbers other than 0 and 1 satisfy these statements.

The so-called cancellation laws for addition and multiplication are as follows:

If $x + a = x + b$, then $a = b$

If $x \cdot a = x \cdot b$, x not zero, then $a = b$

Proof of these laws will not be presented.

Finite and infinite. The set of natural numbers (cardinals with zero omitted) has some peculiar properties. The set of even numbers $\{2, 4, 6, \ldots\}$ is obviously a proper subset of the set $\{1, 2, 3, 4, \ldots\}$. Yet, these two sets may be put into a one-to-one correspondence. As witness:

$$\begin{array}{ccccc} 2 & 4 & 6 & 8 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & \ldots \end{array}$$

Does this mean that these two sets have the same cardinal number? How can a proper subset of a set have the same number as the set? All of our previous discussion has been about sets whose elements we could name or write.

A set is *infinite* if it has a proper subset such that a one-to-one correspondence exists between the elements of the set and the elements of the subset. The set of cardinal numbers $\{0, 1, 2, 3, \ldots, n, \ldots\}$ is an infinite set. Sets which are not infinite are called *finite*.

NUMBER SYSTEMS

Number operations may be defined in several ways. We have so far defined two operations on cardinal numbers. In general, a number operation associates with two numbers a third number. This is called a *binary* operation. Addition and multiplication are binary operations. There may be *unitary* operations. *Can you give an example?*

A *number system* is defined to be a set of objects for which two binary operations have been defined such that one operation is commutative and associative and the second operation is commutative, associative, and distributive with respect to the first operation.

The cardinal numbers and our operations of addition and multiplication form a number system. This, however, is only one example of a number system. The remainder of this chapter is devoted to expanding the cardinal numbers to other sets of numbers which satisfy this definition of a number system.

Remember that there are many ways to write cardinal numbers. Roman numerals or other systems of numeration may be used. Each of these systems may be considered a separate number system with addition and multiplication defined for that system. We say that these systems have the same *structure*. This means there is a *mapping* of one system into the other that puts the numbers in one system into a one-to-one correspondence with the numbers of the other, and in this mapping number operations are preserved. This means that if a and b are numbers in one system, then there is a mapping that makes them correspond to a' and b' in the other system and, furthermore, $a + b$ maps into $a' + b'$ and $a \cdot b$ maps into $a' \cdot b'$.

Such number systems are called *isomorphic* and are said to have the same structure. We will expand the cardinal numbers into other number systems which will have proper subsets isomorphic to the cardinal numbers.

C. Extending the Number System

Our cardinal number system, the set $\{0, 1, 2, 3, \ldots\}$, along with the operations of addition and multiplication, has the property of *closure*. This means that if x and y are cardinal numbers, then $x + y$ and $x \cdot y$ are always cardinal numbers. So far, in our more or less formal development of the cardinal number system, the operations of subtraction and division have not been mentioned.

SUBTRACTION OF CARDINAL NUMBERS

Consider the statement $a + x = b$ where a and b are cardinal numbers and x is a variable whose domain is the set of cardinal numbers. Is there an x which will make this a true statement for any numbers a and b? The statement $2 + x = 7$ has the solution $x = 5$.

But, what about $7 + x = 4$? Is there any number which added to 7 gives 4? Obviously not among the cardinal numbers.

We define subtraction in terms of addition. Thus, $x = b - a$ means $a + x = b$. For example: $5 = 7 - 2$ means $2 + 5 = 7$. Subtraction *is the inverse of addition*. The term "$b - a$" is read "b minus a." It is sometimes called a *difference*.

Now it may be seen that our cardinal numbers are not *closed* under the operation of subtraction. There are many natural numbers c and d such that $d - c$ has no answer. To supply an answer for all such numbers, we need to extend our cardinal number system to the system of *integers*.

THE INTEGERS

We have used the concept "ordered pair" in connection with elements of sets. Now, we can use this concept on cardinal numbers to define *integers*. These are the positive and negative numbers and zero. An *integer* is defined to be an ordered pair of cardinal numbers. Examples are $(1, 3)$, $(0, 4)$, $(8, 6)$, and $(247, 0)$.

We can define an equality or order relationship among these ordered pairs. The integer (a, b) is defined to be equal to (c, d) if, and only if, $a + d = b + c$. Note that we have already defined addition of cardinal numbers. You may notice at this point that we have families of equal integers. For example, $(3, 1)$, $(4, 2)$, $(5, 3)$, $(6, 4)$, and so on. Since $3 + 4 = 1 + 6$, then $(3, 1) = (6, 4)$. This family of equal integers contains a member that has zero as one of the ordered pair of cardinal numbers, the integer $(2, 0)$. In fact, every family of equal integers has a member in which one of the two cardinal numbers is zero. These members may be thought of as the standard form for the family of equal integers. In general, if an integer may be put in the form $(k, a + k)$ or $(a + k, k)$, where a and k are cardinal numbers, then the standard forms are $(0, a)$ and $(a, 0)$ respectively. *Can you show equality?*

Integers may be ordered by the following definition. An integer (a, b) is greater than, or less than, an integer (c, d) according as $a + d$ is greater than or less than $b + c$. Is $(3, 1) > (1, 3)$? Notice that $3 + 3 > 1 + 1$, therefore $(3, 1) > (1, 3)$. The standard form for $(3, 1)$ is $(2, 0)$, and for $(1, 3)$ the standard form is $(0, 2)$.

We may now determine that we have an ordered set of integers in standard form as follows:

$$\ldots < (0, 2) < (0, 1) < (0, 0) < (1, 0) < (2, 0) < \ldots$$

This ordered set of integers in standard form may be given the symbols for positive and negative integers and zero as follows:

$$\begin{array}{cccccccc} \ldots, & (0, 2), & (0, 1), & (0, 0), & (1, 0), & (2, 0), & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \ldots, & -2, & -1, & 0, & 1, & 2, & \ldots \end{array}$$

The integers greater than zero are called *positive integers*; those less than zero are called *negative integers.*

ADDITION OF INTEGERS

Addition of integers may be defined as follows:

$$(a, b) + (c, d) = (a + c, b + d)$$

Why have this definition? Such a definition makes addition commutative and associative and helps to form a new enlarged number system. Does the form of (a, b) and (c, d) affect the definition of addition? Let's try an example.

$$(4, 1) + (7, 2) = (4 + 7, 1 + 2)$$

Upon adding the cardinals we get $(11, 3)$, which in standard form is $(8, 0)$. If we write $(4, 1)$ and $(7, 2)$ in standard form we have

$$(3, 0) + (5, 0) = (3 + 5, 0 + 0) = (8, 0)$$

Using our new symbols, we have

$$(+3) + (+5) = +8$$

Now try $(4, 1)$ and $(3, 8)$. In standard form these are $(3, 0)$ and $(0, 5)$ and, upon adding, we have

$$(3, 0) + (0, 5) = (3 + 0, 0 + 5) = (3, 5)$$

In standard form $(3, 5) = (0, 2)$, and in our new symbols this is -2. Suppose we change these integers to the new symbols:

$$(4, 1) = (3, 0) = +3$$

$$(3, 8) = (0, 5) = -5$$

Then we have $(+3) + (-5) = -2$. Parentheses are used so positive and negative signs will not be confused with addition or subtraction symbols.

Laws of addition. The integers obey the commutative and associative laws of addition. The commutative law states that $x + y = y + x$,

where x and y are variables whose domains are the set of integers. Let (a, b) and (c, d) be ordered pairs of cardinal numbers, that is, integers. Then we need to show that

$$(a, b) + (c, d) = (c, d) + (a, b)$$

Now, by definition, $(a, b) + (c, d) = (a + c, b + d)$

and $$(a + c, b + d) = (c + a, d + b)$$

since addition of cardinal numbers is commutative. However, $(c, d) + (a, b)$ also equals $(c + a, d + b)$, and the law is established.

The associative law states that $(x + y) + z = x + (y + z)$, where x, y, and z are variables whose domains are the set of integers. Let (a, b), (c, d), and (e, f) be any three integers. We need to prove that

$$((a, b) + (c, d)) + (e, f) = (a, b) + ((c, d) + (e, f))$$

Now, by definition,

$$((a, b) + (c, d)) + (e, f) = (a + c, b + d) + (e, f)$$

and $$\begin{aligned}(a + c, b + d) + (e, f) &= ((a + c) + e, (b + d) + f) \\ &= (a + (c + e), b + (d + f))\end{aligned}$$

because addition of cardinal numbers is associative. However,

$$(a, b) + ((c, d) + (e, f)) = (a + (c + e), b + (d + f))$$

and the law has been proved.

MULTIPLICATION OF INTEGERS

Multiplication of integers is defined as follows:

$$(a, b) \cdot (c, d) = ((a \cdot c) + (b \cdot d), (a \cdot d) + (b \cdot c))$$

where (a, b) and (c, d) are ordered pairs of cardinal numbers, that is, integers. Why this definition? You have noticed that the positive integers and zero correspond to our cardinal numbers. The foregoing definition then corresponds to the definition of multiplication of cardinal numbers. We desired a definition of multiplication for all the integers that would obey the three laws of multiplication. Note that the definition is the same whether (a, b) and (c, d) are in standard form or not.

As an illustration, find the product of $(4, 2)$ and $(7, 3)$. By definition, this product is $(28 + 6, 12 + 14)$ which is $(34, 26)$. Now put the two

integers in standard form. Then, $(4, 2) = (2, 0)$ and $(7, 3) = (4, 0)$ and we have the product

$$(2, 0) \cdot (4, 0) = (8, 0)$$

Can you show this is true by the definition? Using our symbols for integers, we have $(+2) \cdot (+4) = +8$, which corresponds to the product of the cardinal numbers 2 and 4. Notice that the original product $(34, 26)$ in standard form is $(8, 0)$.

Now find the product of $(5, 1)$ and $(3, 6)$. By definition,

$$(5, 1) \cdot (3, 6) = (15 + 6, 30 + 3) = (21, 33)$$

Now in standard form, we have

$$(4, 0) \cdot (0, 3) = (0 + 0, 12 + 0) = (0, 12)$$

We see that $(21, 33)$ in standard form is $(0, 12)$. Using the symbols for integers, this would be written $(+4) \cdot (-3) = -12$. Note that the product of a positive integer and a negative integer is a negative integer.

One other illustration of multiplication is revealing. The product of $(1, 3)$ and $(4, 9)$ is

$$(1, 3) \cdot (4, 9) = (4 + 27, 9 + 12) = (31, 21)$$

With the integers in standard form, this would be written

$$(0, 2) \cdot (0, 5) = (0 + 10, 0 + 0) = (10, 0)$$

Writing this statement with the symbols for integers, we have

$$(-2) \cdot (-5) = +10.$$

Note that the product of a pair of negative integers is a positive integer.

Students have long experienced difficulty in understanding why the product of two negative integers is a positive integer, or why the product of a positive integer and a negative integer is a negative integer. Notice that we have not proved these statements. They are simply true because the definition of multiplication of integers makes them so. The usual rules for "signs" in addition and multiplication of integers come from our definitions of these operations.

Laws of multiplication. The integers obey the commutative and associative laws of multiplication as do the cardinals. The commutative law states that $x \cdot y = y \cdot x$, where x and y are variables whose domains are the set of integers. (Where letters are used to represent numbers,

multiplication may be indicated by writing the letters adjacent with no sign. Thus $x \cdot y = y \cdot x$ may be written $xy = yx$.) Let (a, b) and (c, d) be integers. Then we need to prove that

$$(a, b) \cdot (c, d) = (c, d) \cdot (a, b)$$

Now, by definition,

$$(a, b) \cdot (c, d) = (ac + bd, ad + bc)$$

and since addition and multiplication of cardinal numbers are both commutative, we have

$$\begin{aligned}(ac + bd, ad + bc) &= (ca + db, da + cb)\\ &= (ca + db, cb + da)\end{aligned}$$

We see, however, that

$$(c, d) \cdot (a, b) = (ca + db, cb + da)$$

by definition. The law has been demonstrated.

The associative law states that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ where x, y, and z are variables whose domains are the set of integers. Let (a, b), (c, d), and (e, f) be integers. We shall prove that

$$((a, b) \cdot (c, d)) \cdot (e, f) = (a, b) \cdot ((c, d) \cdot (e, f))$$

If we can, by following already established rules, change one side of the foregoing statement into the other side, the law will be established. We may write

$$((a, b) \cdot (c, d)) \cdot (e, f) = (ac + bd, ad + bc) \cdot (e, f)$$

which, by definition of multiplication, equals

$$\begin{aligned}&((ac + bd)e + (ad + bc)f, (ac + bd)f + (ad + bc)e)\\ &\qquad = (ace + bde + adf + bcf, acf + bdf + ade + bce)\end{aligned}$$

Now, from this expression, still following established rules, we can "take out" (a, b) to get the desired proof.

$$\begin{aligned}&(a(ce + df) + b(de + cf), a(cf + de) + b(df + ce))\\ &\qquad = (a(ce + df) + b(cf + de), a(cf + de) + b(ce + df))\\ &\qquad = (a, b) \cdot (ce + df, cf + de)\\ &\qquad = (a, b) \cdot ((c, d) \cdot (e, f))\end{aligned}$$

This completes the proof of the law.

The multiplication of integers is distributive with respect to addition. If x, y, and z are variables whose domains are the set of integers, then

$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$

Let (a, b), (c, d), and (e, f) be integers. Then we wish to prove that

$$(a, b) \cdot ((c, d) + (e, f)) = (a, b) \cdot (c, d) + (a, b) \cdot (e, f)$$

Now the left side of this statement is

$$\begin{aligned}(a, b) \cdot (c + e, d + f) &= (a(c + e) + b(d + f), a(d + f) + b(c + e)) \\ &= (ac + ae + bd + bf, ad + af + bc + ef)\end{aligned}$$

but the right side of the preceding expression also can be changed to this identical form by already established rules for cardinal numbers. Therefore, the distributive law is demonstrated.

SPECIAL ELEMENTS AND CONCEPTS

Just as we had the special elements zero and one in the set of cardinal numbers, we have similar elements in the set of integers. In the ordered-pair notation, they would be $(0, 0)$ and $(1, 0)$. In the symbols for integers, they would be 0 and $+1$.

We need to demonstrate the following statements for these special integers:

$$0 + x = x \qquad 0 \cdot x = 0 \quad \text{and} \quad +1 \cdot x = x$$

where x is an integer. Let (a, b) be any integer. Then

$$(0, 0) + (a, b) = (0 + a, 0 + b) = (a, b),$$

and

$$\begin{aligned}(0, 0) \cdot (a, b) &= (0 \cdot a + 0 \cdot b, 0 \cdot b + 0 \cdot a) \\ &= (0, 0)\end{aligned}$$

Also,

$$\begin{aligned}(1, 0) \cdot (a, b) &= (1 \cdot a + 0 \cdot b, 1 \cdot b + 0 \cdot a) \\ &= (a, b)\end{aligned}$$

The statements have been demonstrated.

The negative of an integer. Let x be an integer. Then the integer x' is the negative of x if $x + x' = 0$. If (a, b) is any integer, then the negative of (a, b) is (b, a). The proof is as follows:

$$(a, b) + (b, a) = (a + b, b + a)$$

but since $a + b = b + a$, the standard form for $(a + b, b + a)$ is $(0, 0)$. The negative of an integer x will be denoted by $-x$. Note that the negative of $(6, 3)$ is $(3, 6)$. In standard form, these are $(3, 0)$ and $(0, 3)$, or, in our symbols for integers, $+3$ and -3. Hence $(+3) + (-3) = 0$. Also note that the negative of -3 is $+3$.

Now if x is an integer, then $(-1) \cdot x = -x$. This means that any integer multiplied by negative one is the negative of the integer. Let (a, b) be any integer. Then

$$\begin{aligned} -1 \cdot (a, b) &= (0, 1) \cdot (a, b) \\ &= (0 \cdot a + 1 \cdot b, 0 \cdot b + 1 \cdot a) \\ &= (b, a) \end{aligned}$$

We have previously shown that (b, a) is the negative of (a, b).

Subtraction of integers. The integers were formed in an effort to find a set of numbers in which subtraction would always be possible. Previously, it was shown that subtraction was not always possible with the cardinal numbers. Thus $x = b - a$, which means find the number x such that $a + x = b$, does not always have a solution. In the set of integers, this definition of subtraction does always have a solution.

We wish to prove that if a and b are any integers, then there is an integer x such that $a + x = b$. (In other words, $x = b - a$, where "$-$" denotes subtraction.) (Note that we also use this symbol for the negative of an integer.) We will prove that if $b + (-a)$ is substituted for x, the statement $a + x = b$ is true. Now,

$$\begin{aligned} a + (b + (-a)) &= (b + (-a)) + a \\ &= b + ((-a) + a) \\ &= b + 0 \\ &= b \end{aligned}$$

since addition of integers is commutative, associative, and the sum of an integer and its negative is zero. Therefore, subtraction is always possible with integers.

What is $5 - 8$? This expression means $(+5) - (+8)$. We may omit the sign in front of positive integers, but not in front of negative integers. The symbol "$-$" here means subtract. According to our proof that subtraction is always possible $(+5) - (+8)$ is equal to $(+5) + (-8)$. *What is the standard form for the integer* $(+5) + (-8)$? Write these integers as $(5, 0) + (0, 8)$. Add to get $(5 + 0, 0 + 8) = (5, 8) = (0, 3)$, which is the integer -3. We say that $5 - 8 = -3$. Our conventional way of writing integers or "signed" numbers is confusing. In the statement $5 - 8 = -3$, the first "$-$" means subtraction; the second "$-$" means negative of $+3$. This sort of usage has led to our usual "rules" for operations on signed numbers.

INTEGERS FORM A NUMBER SYSTEM

We have now defined a set of objects called *integers* and two binary operations called ***addition*** and ***multiplication*** which are commutative and associative, with multiplication being distributive with respect to addition. Thus a new enlarged number system has been developed out of the cardinal numbers. There is, however, a subset of the integers that is isomorphic to the cardinals. The set of positive integers and zero may be put into a one-to-one correspondence with the cardinal numbers as follows:

$$\begin{array}{ccccc} 0, & +1, & +2, & +3, & +4, \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0, & 1, & 2, & 3, & 4, \ldots \end{array}$$

Also, the sum and product of two integers corresponds to the sum and product of the two corresponding cardinals. Some writers view zero and the positive integers and the cardinals as being the same set of objects. They look upon the integers as having the cardinals as a proper subset. Other writers are careful to point out that the set of cardinals and the set of non-negative integers are not the same, but have this special relationship called *isomorphism.*

D. The Rational Numbers

Our system of integers has the property of closure under the operations of addition, multiplication, and subtraction. This means that if x and y are any two integers, then $x + y$, $x \cdot y$, and $x - y$ are integers. In our logical development of numbers, we have not yet defined division. The operation of division will be defined in terms of multiplication.

DIVISION OF NUMBERS

Consider the statement $ax = b$ where a and b are integers and x is a variable whose domain is the set of integers. We may ask whether there exists an x which makes this a true statement for any a and b. The statement $3x = 12$ has a solution $x = 4$, because $3 \cdot 4 = 12$. How about $3x = 14$? What number multiplied by 3 gives 14? Obviously none, since $3 \cdot 4 = 12$, and $3 \cdot 5 = 15$.

Division may be defined as the inverse of multiplication. Thus $x = b \div a$ means $a \cdot x = b$, where a cannot be 0. For example,

$3 = 12 \div 4$ means that $3 \cdot 4 = 12$. The term "$b \div a$" is read "b divided by a." It is commonly called a *quotient.*

It may be seen that our set of integers is not *closed* under the operation of division. This simply means that there are integers a and b such that $b \div a$ has no "answer." Can we extend the set of integers as we did the set of cardinals to get a new set in which division is always possible? The answer is yes. The device used is similar. We may use the idea of ordered pairs of integers to define an enlarged set of objects called *rationals.* We may then define the operations of addition and multiplication of rationals and show that they obey our five laws. Therefore, we have a new number system called the *rational* number system. This is our familiar system of common fractions.

THE RATIONALS

Consider the set of all ordered pairs of integers. These may be written (a, b), $\frac{a}{b}$, or a/b, with the requirement that b is never 0. We define a *rational* to be an ordered pair of integers. Thus $(3, 4)$, $(1, 2)$, and $(-6, 2)$, which may be written $3/4$, $1/2$, and $-6/2$, are rationals.

Two rationals a/b and c/d are said to be equal, if, and only if, $ad = bc$. Once again, we have families of equal ordered pairs. For example, $3/4 = 9/12$ since $3 \cdot 12 = 4 \cdot 9$. Also, $3/4 = 6/8 = 9/12 = 12/16$, and so on. Members of this family may be written in the form $3m/4m$ where m is any integer. For example, let $m = -6$, then,

$$\frac{3 \cdot (-6)}{4 \cdot (-6)} = \frac{-18}{-24}$$

Is this equal to $3/4$? By our definition $(3) \cdot (-24)$ should equal $(4) \cdot (-18)$. Both products are -72; therefore, $-18/-24 = 3/4$. Now our definition of equality may be used to point out some special families of rationals. If a, b, and m are any non-zero integers, then:

$$\frac{am}{bm} = \frac{a}{b}, \quad \text{since} \quad (am)b = (bm)a$$

$$\frac{0}{a} = \frac{0}{1}, \quad \text{since} \quad 0 \cdot 1 = 0 \cdot a \quad \text{for any } a$$

$$\frac{a}{a} = \frac{1}{1}, \quad \text{since} \quad a \cdot 1 = a \cdot 1$$

Also, $$\frac{am}{a} = \frac{m}{1}, \quad \text{since} \quad (am) \cdot 1 = a \cdot m$$

$$\frac{a}{am} = \frac{1}{m}, \quad \text{since} \quad a \cdot m = (am) \cdot 1$$

We may define a standard form for a rational. Since rationals are families of equal ordered pairs of integers, this means selecting one member of each family to represent the family. Because integers, other than zero, are either positive or negative, we have two cases. Consider the rational number in which the two integers in the ordered pair have different signs. For example, $-2/+3$. Note that

$$\frac{-2}{+3} = \frac{+2}{-3}, \quad \text{since} \quad (-2)(-3) = (+3)(+2)$$

(Note that parentheses written together with no sign between them means multiply. Thus (5)(4) means $5 \cdot 4$ or 5×4.) The foregoing equality is true for all such rationals. We shall choose as our standard form the ordered pair in which the second integer is positive. The other ease is one in which the two integers in the ordered pair have like signs; That is, both are positive or both are negative. Again, we may show that $-2/-3$ and $+2/+3$ belong to the same family. We have

$$\frac{-2}{-3} = \frac{+2}{+3}, \quad \text{since} \quad (-2)(+3) = (-3)(+2)$$

We shall choose the ordered pair for which both integers are positive as our standard form. If any rational can be put in the form am/bm where m is any integer, then we will choose the form a/b as our standard form.

If rationals are put in standard form, we may define an order relation. Let a/b and c/d be rationals in standard form. Then

$$\frac{a}{b} > \frac{c}{d}, \quad \text{if} \quad ad > bc$$

and $$\frac{a}{b} < \frac{c}{d}, \quad \text{if} \quad ad < bc$$

Remember that any positive integer is greater than any negative integer.

OPERATIONS WITH RATIONALS

Let a/b and c/d be rationals. Then addition is defined as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$$

For example,

$$\frac{1}{2} + \frac{3}{5} = \frac{(1 \cdot 5 + 2 \cdot 3)}{2 \cdot 5} = \frac{11}{10}$$

By methods similar to those used for the integers, we may show that addition of rationals is commutative and associative.

With subtraction defined as the inverse of addition, we find that subtraction of rationals is always possible and is defined as follows:

$$\frac{a}{b} - \frac{c}{d} = \frac{(ad - bc)}{bd}$$

Multiplication of rationals is defined by the following statement:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

For example,

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{3 \cdot 2}{4 \cdot 3} = \frac{6}{12}$$

The standard form for $\frac{6}{12}$ is $\frac{1}{2}$, since

$$\frac{6}{12} = \frac{1 \cdot 6}{2 \cdot 6} = \frac{1}{2}$$

We can prove that multiplication of rationals is commutative, associative, and distributive with respect to addition.

With division defined as the inverse of multiplication, we may show that division is always possible with rationals. Let a/b and c/d be rationals. Then $a/b \div c/d = x/y$ where x/y is a rational. This means that

$$\frac{a}{b} = \frac{c}{d} \cdot \frac{x}{y} = \frac{cx}{dy}$$

By our definition of equality of rationals, we have $ady = bcx$. If we let $y = bc$ and $x = ad$, the last statement becomes an identity. Thus,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad}{bc}$$

This definition of division of rationals as the inverse of multiplication is the source of our rule to invert the divisor and multiply in order to divide common fractions. As an example of the division of rationals:

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \cdot \frac{3}{2} = \frac{3 \cdot 3}{4 \cdot 2} = \frac{9}{8}$$

SPECIAL ELEMENTS AND CONCEPTS

The cardinals and the integers have special elements 0 and 1. Are such elements among the rationals? Consider the rational a/a, which may be put in the standard form $1/1$. Let a/b be any rational. Then

$$\frac{1}{1} \cdot \frac{a}{b} = \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b}$$

because of the properties of the integer 1. The rational $1/1$ has the same property as the one-element for cardinals and integers.

Now we need to show that the rational $0/a$, standard form $0/1$, has the properties of the zero element, namely,

$$\frac{0}{1} + \frac{a}{b} = \frac{a}{b}$$

$$\frac{0}{1} \cdot \frac{a}{b} = \frac{0}{1}$$

First,
$$\frac{0}{1} + \frac{a}{b} = \frac{(0 \cdot b + 1 \cdot a)}{1 \cdot b} = \frac{a}{b}$$

because of the properties of the integers 0 and 1.

Second,
$$\frac{0}{1} \cdot \frac{a}{b} = \frac{0 \cdot a}{1 \cdot b} = \frac{0}{b}$$

but $0/b$ in standard form is $0/1$, and the two properties have been demonstrated.

We may show that every rational has a negative. You will recall that the sum of an integer and its negative is zero. The same definition will be used for rationals. If a/b is a rational, then $-a/b$ is its negative, since

$$\begin{aligned}\frac{a}{b} + \frac{-a}{b} &= \frac{(a \cdot b - b \cdot a)}{b \cdot b} \\ &= \frac{0}{b \cdot b} \\ &= \frac{0}{1}\end{aligned}$$

the zero element for the rationals. Note that $-a$ is the negative of the integer a. This was defined for integers earlier. We are making use of the properties of integers previously defined and demonstrated.

Let a/b be a rational; then,

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{ab}{ab} = \frac{1}{1}$$

The rational b/a is called the *reciprocal* of a/b. The product of a rational and its reciprocal is the one element $1/1$. The zero element $0/1$ has no reciprocal since $1/0$ is not a rational. You will recall that the second integer in the ordered pair of integers could not be zero. *Can you show that the one element among the rationals is its own reciprocal?* Note, too, that the definition of reciprocal of a rational makes the definition of division simple. To divide by a rational, multiply by its reciprocal.

We may now demonstrate a peculiar characteristic of rationals. Between any two rationals there is always another rational. This means that if a/b and c/d are rationals such that $a/b < c/d$, then there exists a rational greater than a/b and less than c/d. We can prove that

$$\frac{a + c}{b + d}$$

is such a rational. This means that

$$\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}$$

Now $$\frac{a}{b} < \frac{a + c}{b + d}$$

since $$a(b + d) < b(a + c)$$

This is true because $$ab + ad < ab + bc$$

since $ad < bc$. *Why is $ad < bc$?*

Also, $$\frac{a + c}{b + d} < \frac{c}{d}$$

since $$(a + c)d < (b + d)c$$

This is true because $$ad + cd < bc + cd$$

since $ad < bc$. This means that given a rational a/b, there is no next rational. If you select a rational c/d, no matter how close to a/b, there are always more rationals in between. In fact, between any two rationals there is an infinite set of rationals.

RATIONALS A NUMBER SYSTEM

Our rationals and the operations of addition and multiplication meet the requirements for a number system. We have enlarged on our system of integers to get a new number system. There is, however, a subset of the rational number system isomorphic to the system of integers. This subset is the set of rationals of the form $a/1$ where a is any integer, negative, zero, or positive. This subset may be ordered and put into a one-to-one correspondence with the integers as follows:

$$\begin{array}{cccccccccc} \ldots, & \frac{-3}{1}, & \frac{-2}{1}, & \frac{-1}{1}, & \frac{0}{1}, & \frac{1}{1}, & \frac{2}{1}, & \frac{3}{1}, & \ldots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ \ldots, & -3, & -2, & -1, & 0, & 1, & 2, & 3, & \ldots \end{array}$$

It may be easily shown that the sum and product of two of the foregoing rationals correspond to the sum and product of the integers corresponding to the two rationals. We have

$$\frac{a}{1} + \frac{b}{1} = \frac{a+b}{1}$$

and

$$\frac{a}{1} \cdot \frac{b}{1} = \frac{a \cdot b}{1}$$

which corresponds to $a + b$ and $a \cdot b$ among the integers. Thus there is a subset of the system of rational numbers which is isomorphic to the system of integers.

From now on, we will use the symbols for integers to represent those rationals which are isomorphic to the integers. This means that we will write $0/1$ as 0, $1/1$ as 1, $-4/1$ as -4, and so on.

E. The Real Numbers

The rational number system seems to fill all the requirements which we need for numbers. The system is closed under the four fundamental operations. (Remember that division by zero is excluded.) This means that if we perform any of the operations on two rational numbers, we always get another rational number. There is always an "answer." Are there other occasions where an answer is not available? We will exhibit some instances in which the rational numbers do not give an answer and indicate briefly how the rational numbers may be extended. This time the extension cannot

be accomplished with ordered pairs of rationals. The concept of the real number system is much more difficult and subtle than that of our previous number systems. A rigorous development of the real numbers is beyond the scope of this book. A simplified version of only one of several ways of developing the real numbers will be presented.

EXPRESSIONS AND EQUATIONS

An *expression* is a number or a variable whose domain is the set of numbers. The set of numbers may be the cardinals, the integers, or the rationals. Most generally, we understand the rationals to be the set unless otherwise specified. Also, we will define the sum or product of expressions to be expressions.

When a number is multiplied by itself several times, we may use a short-cut way of writing it. For example, $3 \cdot 3$ may be written 3^2; $x \cdot x \cdot x$ may be written x^3; or $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ may be written $(\frac{1}{2})^4$. These are called "3 squared," "x cubed," and "$\frac{1}{2}$ to the fourth power." The power to which the number is raised is called an *exponent.* The inverse of raising to a power is extracting a root. For example, $2^2 = 4$, whereas $\sqrt{4} = 2$. We say "the square root of 4 is 2," meaning that 2 squared is 4. Likewise, "the cube root of 64 is 4" means that 4 cubed is 64. This will shorten our notation when we have repeated multiplication.

Consider two expressions containing rational numbers and/or a variable whose domain is the set of rational numbers. A statement that the two expressions are equal is called an *equation*. The rational numbers which make the statement true when substituted for the variable are called the *solution set*. Examples of equations are

$x + 5 = 8$	solution set is $\{3\}$
$x - 3 = 2$	solution set is $\{5\}$
$x^2 = 25$	solution set is $\{-5, 5\}$
$3 \cdot x = 12$	solution set is $\{4\}$
$\frac{1}{2}y^2 = \frac{1}{32}$	solution set is $\{\frac{1}{4}, -\frac{1}{4}\}$
$x^2 = 2$	solution set is ?

What number multiplied by itself gives 2? We say the $\sqrt{2}$. Our variable x, however, must be a rational number. Now is the $\sqrt{2}$ a rational number? If it isn't, we now have found an occasion where the rational numbers do not provide us with an "answer."

IRRATIONAL NUMBERS

Do numbers beyond the rationals exist? Is it our job to discover these numbers, or do we simply define or "construct" objects which may be called numbers? The most elementary concept of number, that of plurality or manyness, has already been enlarged upon. Does a rational number tell how many? Does a negative integer tell how many? No? Then what do these numbers tell? Physical interpretations of these numbers may be made. From our elementary concepts of manyness and sets, however, we have defined or "constructed" other abstract sets which obey certain rules. These, then, are our numbers. We should then feel free to go ahead and define other objects and call them numbers if we can define operations that will be in accord with our general definition of a number system.

We may now show that $\sqrt{2}$ is not a rational number. This proof was discovered by the Greeks in their efforts to find the diagonal of a unit square. To their surprise, the diagonal could not be expressed as a rational number. First, we need to look at odd and even positive integers. Note that an integer a is even if, and only if, $a = 2m$ where m is an integer and is odd if, and only if, $a = 2m + 1$. We may show that the square of an even integer is even, and the square of an odd integer is odd. *Can you show this?* Now suppose that the $\sqrt{2}$ is some rational number a/b. Note that not both a and b are even. Then,

$$\left(\frac{a}{b}\right)^2 = 2$$

$$\frac{a^2}{b^2} = 2 \qquad \text{or} \qquad a^2 = 2b^2$$

Now since a^2 is equal to the product of 2 and the integer b^2, then a^2 is an even integer and, hence, a is an even integer. Let $a = 2k$, where k is an integer. Then we have

$$(2k)^2 = 2b^2$$

$$4k^2 = 2b^2 \qquad \text{or} \qquad 2k^2 = b^2$$

Therefore, we see that b is an even integer. We have shown that both a and b are even which contradicts our assumption that there is a rational whose square is 2.

It may be shown that many equations do not have solutions in the set of rational numbers. If solutions exist at all, they must exist in some

extended number system. In other words, we need to enlarge our definition of number so that other objects of our conception may be called numbers. Another example is the ratio of the circumference of a circle to its diameter, commonly denoted by the symbol π. If the diameter of a circle is a rational number, then the circumference is not a rational.

DECIMALS

We have previously demonstrated our positional system of numeration. This means that any positive integer may be written in the form

$$a_n(10^n) + a_{n-1}(10^{n-1}) + \ldots + a_2(10^2) + a_1(10) + a_0$$

where $a_0 \ldots a_n$ are one of the digits $0 \ldots 9$. The decimal system of notation extends this concept to positive numbers less than 1. (Since elementary teachers are not much concerned with negative numbers, our discussion of decimals will be restricted to non-negative numbers. With the proper explanation, everything in this section will also apply to negative numbers.)

A finite decimal represents a rational whose denominator is some power of 10. Thus $\frac{3}{10}$ is written as .3, $\frac{75}{100}$ as .75, and $\frac{237}{1000}$ as .237. Notice that $\frac{237}{1000}$ may be written as

$$\frac{2}{10} + \frac{3}{100} + \frac{7}{1000} \qquad \text{or} \qquad 2\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right) + 7\left(\frac{1}{10^3}\right)$$

Thus,

$$.237 = 2(10^{-1}) + 3(10^{-2}) + 7(10^{-3})$$

if we use the notation

$$\frac{1}{10^n} = 10^{-n}$$

Now what about rationals whose denominators are not some power of 10? (This implies that they cannot be changed to an equal form which has a power of 10 for the denominator.) If we use our familiar long division form we can secure the digits that make up the decimal form. This implies that we already know what decimals are and how to operate on them. These digits, however, may be found by the so-called division algorithm without using decimal notation. The division algorithm will not be considered in this book.

Rationals which cannot be expressed as finite decimals may be expressed as *infinite* decimals. This means that no matter how many

decimal places are computed or determined, we never get a zero. For example,

$$\tfrac{1}{3} = .3333\ldots$$
$$\tfrac{1}{7} = .142857142857142857\ldots$$

For 1/3, note that we have "3" repeated endlessly, and for 1/7, we have "142857" repeated endlessly. If, for the rationals which are expressed by finite decimals, we consider the zeros repeated without end, we can state that all rationals may be expressed by infinite repeating decimals.

Now given any rational, we can find the infinite repeating decimal which represents it. If we are given any infinite repeating decimal, can we find the rational it represents? We illustrate the method. Given the infinite repeating decimal .181818 . . . , what rational does it represent? Let

$$x = .181818\ldots$$

Then
$$10^2x = 18.1818\ldots$$

and
$$10^2x - x = 18.1818\ldots - .1818\ldots$$
$$= 18$$
$$x(10^2 - 1) = 18$$
$$x = \frac{18}{99} \quad \text{or} \quad \frac{2}{11}$$

We assume that the infinite repeating decimal cancels upon subtraction. This method leads us to some other findings about decimals. We know that $\tfrac{1}{2} = .5$ or $.5000\ldots$, but what rational is represented by $.4999\ldots$? Using the method just illustrated, we have

$$x = .4999\ldots$$
$$10x = 4.999\ldots$$
$$100x = 49.999\ldots$$
$$100x - 10x = 49.999\ldots - 4.999\ldots$$
$$x(100 - 10) = 45$$
$$x = \frac{45}{90} = \frac{1}{2}$$

Therefore, an infinite decimal which has nines repeating endlessly may be replaced by a decimal with all zeros. Thus,

$$.4999\ldots = .5000\ldots$$
$$2.6999\ldots = 2.7000\ldots$$
$$1.33999\ldots = 1.34000\ldots$$

Every finite decimal and every infinite repeating decimal represents some rational number, and every rational number may be so represented. Now what about infinite decimals that do not repeat? Some familiar examples are

$$\sqrt{2} = 1.4142\ldots$$

$$\pi = 3.1415\ldots$$

Another example frequently cited is the infinite decimal

$$.1010010001\ldots,$$

where each "1" is followed by a set of zeros containing one more zero than the set of zeros preceding the "1." Are these numbers? So far, we would say no.

REAL NUMBERS

We define real numbers as the set of all infinite decimals.

$$\pm\ldots a_3a_2a_1 . b_1b_2b_3\ldots$$

where the a's and b's are the digits $0, \ldots, 9$. Two real numbers are equal if, and only if, they have the same sign and if they have identical digits in each position. If any infinite decimal repeats nines endlessly, then we agree to replace the nines with zeros and increase the digit preceding the first of the repeating nines by one. For example, $29.66999\ldots$ would be replaced by $29.67000\ldots$.

Now let x be any real number $M . b_1b_2b_3\ldots$ and y be any real number $N . c_1c_2c_3\ldots$, when M and N are integers. We say that x and y are equal if $M = N, b_1 = c_1, b_2 = c_2, b_3 = c_3\ldots$.

Operation on decimals. We may define addition and multiplication of real numbers. To add infinite decimals x and y, set up the following sequences of finite decimals:

$$M, \quad M . b_1, \quad M . b_1b_2, \quad M . b_1b_2b_3, \ldots$$

and

$$N, \quad N . c_1, \quad N . c_1c_2, \quad N . c_1c_2c_3, \ldots$$

Then set up the following sequence of sums. This is possible since each finite decimal is a rational and addition is defined for rationals.

$$M + N, \quad M . b_1 + N . c_1, \quad M . b_1b_2 + N . c_1c_2, \ldots$$

This process defines a specific infinite decimal, and we can determine the digits as far as are desired.

decimal places are computed or determined, we never get a zero. For example,

$$\tfrac{1}{3} = .3333\ldots$$
$$\tfrac{1}{7} = .142857142857142857\ldots$$

For 1/3, note that we have "3" repeated endlessly, and for 1/7, we have "142857" repeated endlessly. If, for the rationals which are expressed by finite decimals, we consider the zeros repeated without end, we can state that all rationals may be expressed by infinite repeating decimals.

Now given any rational, we can find the infinite repeating decimal which represents it. If we are given any infinite repeating decimal, can we find the rational it represents? We illustrate the method. Given the infinite repeating decimal .181818..., what rational does it represent? Let

$$x = .181818\ldots$$

Then
$$10^2x = 18.1818\ldots$$

and
$$\begin{aligned} 10^2x - x &= 18.1818\ldots - .1818\ldots \\ &= 18 \\ x(10^2 - 1) &= 18 \\ x &= \frac{18}{99} \quad \text{or} \quad \frac{2}{11} \end{aligned}$$

We assume that the infinite repeating decimal cancels upon subtraction. This method leads us to some other findings about decimals. We know that $\frac{1}{2} = .5$ or $.5000\ldots$, but what rational is represented by $.4999\ldots$? Using the method just illustrated, we have

$$\begin{aligned} x &= .4999\ldots \\ 10x &= 4.999\ldots \\ 100x &= 49.999\ldots \\ 100x - 10x &= 49.999\ldots - 4.999\ldots \\ x(100 - 10) &= 45 \\ x &= \frac{45}{90} = \frac{1}{2} \end{aligned}$$

Therefore, an infinite decimal which has nines repeating endlessly may be replaced by a decimal with all zeros. Thus,

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Operation on decimals. We may define addition and multiplication of real numbers. To add infinite decimals x and y, set up the following sequences of finite decimals:

$$M,\quad M . b_1,\quad M . b_1b_2,\quad M . b_1b_2b_3, \ldots$$

and $$N,\quad N . c_1,\quad N . c_1c_2,\quad N . c_1c_2c_3, \ldots$$

Then set up the following sequence of sums. This is possible since each finite decimal is a rational and addition is defined for rationals.

$$M + N,\quad M . b_1 + N . c_1,\quad M . b_1b_2 + N . c_1c_2, \ldots$$

This process defines a specific infinite decimal, and we can determine the digits as far as are desired.

Multiplication may be defined in a similar way to provide a sequence of products of finite decimals which can be used to determine the digits of an infinite decimal to any number of places.

Since at each point we have rational numbers involved, these operations obey the five laws of addition and multiplication. The real numbers and the operations satisfy our definition of a number system.

We have a zero element 0.000 . . . , a one element .999 . . . or 1.000 . . . which have the usual characteristics of zero and one elements found in our earlier number systems. Also, we may show that every real number has a negative among the real numbers, and every non-zero real number has a reciprocal among the real numbers.

Still have rationals. The real numbers we have defined have a subset isomorphic to the rationals. We have shown how rationals could all be represented by infinite repeating decimals where we consider a succession of zeros as an infinite repeating decimal. Thus, to each rational there is assigned a real number. We have shown that there are real numbers which do not represent a rational. Proof that addition and multiplication of real numbers yield real numbers that continue to correspond to rationals will not be given.

A rigorous discussion of real numbers must somewhere involve the concept of limits. This concept would come at the point where we let an infinite sequence of finite decimals represent an infinite decimal. It involves the idea that, as we go further out in the infinite sequence of rationals, we come closer and closer to some limiting infinite decimal. This infinite decimal is said to be defined by the sequence.

The real number system allows us to find the number which when squared gives 2. In fact, the real number system allows a solution to the equation $x^n = a$ where n is a positive integer, and a is a positive real number. Our need for numbers beyond the rationals was demonstrated when we could find no solution to the equation $x^2 = 2$ among the rationals. In other words, $\sqrt{2}$ does not exist among the rationals. The real numbers supplied an "answer." What about the equation $x^2 = -1$? Is there such a number as $\sqrt{-1}$? What number multiplied by itself gives -1? Any number so far described multiplied by itself gives a positive number. Remember that $(-1)(-1) = +1$. How can any number multiplied by itself give a negative number? The real numbers contain no such number. Another number system must be devised, but this one is beyond the realm of elementary arithmetic. It is called the *complex* number system. A very fitting name, perhaps.

Something to Think About

1. Give a brief review of the meaning formerly given to the terms *logistic*, *algorism*, *arithmetic*, and *fundamental operations*.

2. Explain the distinction between number operations and finding a standard numeral.

3. Reason through the several ways of thinking about subtraction. Use several examples. Which way do you like best? Why?

4. Explain the Russian peasant method of multiplying whole numbers. Demonstrate with an example.

5. How do modern concepts of computation differ from those formerly held?

6. Reason through a proof of one of the laws governing addition and multiplication of whole numbers.

7. How does the system of integers differ from the system of cardinal numbers?

8. Define a number system. Why do we have more than one number system?

9. Reason through a proof of one of the laws governing addition and multiplication of rationals. Show how the definition of rationals is involved in the proof.

10. How do real numbers differ in an essential way from natural numbers, integers and rationals?

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PART THREE / ARITHMETIC IN THE MIDDLE GRADES

Arithmetic in Grade Four

Several things happen to youngsters when they become fourth-graders. The desire to grow up is being realized. This means, among other things, that the study of arithmetic is going to move rather rapidly in the direction of the abstract. Play activities will be correspondingly reduced. Although certain skills, presumably learned in earlier grades, will be retaught, teachers are inclined to expect students to know them. And within the group, an added year of maturity brings a broader range of ability and achievement. The departure from the lower elementary grades in some cases means a change to another building or even perhaps to another school. It is a major transition in the lives of children.

This chapter presents the following topics:

A. Testing and Diagnosis

B. Number and Number Operations in Grade Four

C. Using Arithmetic in Grade Four

D. Problem Solving

E. The Exceptional Learner

A. Testing and Diagnosis

As has been mentioned, diagnosis should be a continuing classroom activity. Much of the work at fourth-grade level is a building or expansion process, and certain basic operations must be clear to the student before he can build. By fourth grade, the evolving nature of arithmetic learning is becoming apparent. Hence, when the teacher assumes that basic processes have been mastered without testing her assumption, she is inviting trouble.

Some of the texts in arithmetic do not introduce formal diagnostic tests until after several weeks of work. The teacher should not, however, construe this as meaning that no diagnosis is done before this time. Since diagnosis is essentially a point of view, the teacher should be on the alert for indications of difficulties from the very beginning. Poor work habits, such as finger-counting, mumbling, or slumping over work in such a way as to conceal it from the teacher, are readily observed. If the student is inclined to repeat certain errors, or to miss exercises when they include a particular number, an analysis of work sheets should bring this to light. For particularly stubborn cases, the technique of listening to the youngster as he works aloud can be most helpful. So a great deal of diagnosis can be carried out without the use of any specific diagnostic instrument.

Obviously, diagnosis is pointless unless the results are used in correctional work. Since this is a very broad field, no attempt will be made to cover it at this time. Most authorities recommend a frontal attack on any difficulties that show up; that is, if the student has trouble with addition exercises involving 7, he should be given extra work on this phase. Frequently, this requires that he use text material designed for an earlier grade level. If this is where he needs to start in order to handle the work successfully, this is where he should start. It is to be hoped that he will not have to stay there very long.

Some commercial diagnostic tests for use in arithmetic have been listed. Also, texts frequently incorporate them. The test made by the teacher is important in diagnosis. To illustrate: many programs stipulate that by the time students enter fourth grade, they should have mastered the 81 addition and 81 subtraction facts (excluding zero facts). If, early in fourth grade, the teacher gives students a test which includes all the addition facts, she should be able to determine how much reteaching is necessary.

Structurally, the prime difference between a diagnostic test and an ordinary achievement test (both teacher-made) is that, in the former,

the teacher makes a special effort to include all of the operations under study. If a process which includes other manipulative skills (such as long division, which includes multiplication and subtraction) is to be diagnosed, all the incidental skills should be tested separately. Obviously, too, checking such a test would involve searching for patterns of errors, not just a scoring of "number right" or "number wrong."

It is worthy of emphasis, however, that diagnosis is not something that happens on Tuesdays or Wednesdays. It is not something that occurs when students take a particular type of test or work on a certain exercise. It should be as continuous as instruction, since without it, in the case of many students, instruction may be ineffective.

B. Number and Number Operations in Grade Four

Since much of arithmetic is built around operations on whole numbers, these receive attention at each grade level. And since it is never safe to assume that a particular operation was mastered earlier, most teachers begin number operations each year with a quick reteaching of the basic facts. Then, new concepts are presented.

UNDERSTANDING THE NUMBER SYSTEM

Some teachers take the position that if students are clear on such terms as place value and place holder, they understand the number system. Although it is true that these are fundamental, there are many other aspects to understanding the number system.

Large numbers. From the first grade, students have moved steadily toward using larger numbers, with much attention being given to reading, writing, and understanding numbers in the hundreds or thousands. Fourth-grade programs provide further work with larger numbers.

Some programs go to numbers written with five digits; others go to six or even seven digits. Generally, however, the approach is the same in these programs. It should be made clear to students that place value is just as vital in a seven-place numeral as it was in a two-place numeral. Obviously, the place-value pockets are of little help here, since this device becomes cumbersome with more than two- or three-place numerals.

Many teachers like to approach the study of large numbers analytically. They show, for example, that there are 10 hundreds in a thousand, 100 hundreds in ten thousand, 1000 hundreds in a hundred thousand, and so on.

Teachers frequently have trouble finding real-life situations in which large numbers are used. One teacher used the heights of various mountains as a meaningful approach to the study of five-place numbers. Others use such quantities as the populations of various states or cities. This, of course, gives an opportunity to include the students' own state or city in such work.

One approach that works well is to use some facts from astronomy. It is not unusual to have a fourth-grade science book point out that the sun is 93,000,000 miles from the earth. This gives the teacher an excellent opportunity to provide work with large numbers. The teacher should select data carefully, however, so that the numbers used can be made meaningful to the class.

It would probably be rather futile to try to help your class to visualize numbers written with five, six, or seven digits. Can you do it? It is important, however, that the youngsters understand such numbers structurally. Further, some students become so fascinated with reading big numbers that they never get around to such tasks as writing them or using then in fundamental operations. All phases of work with these numbers should be carried out, with special effort to show the relationships between large numbers and the more familiar small ones.

Extending place-value concepts. Working with large numbers gives teachers a real opportunity to reteach the concepts of place value. One common method is to have on the chalkboard a chart (Fig. 7.1) showing the place value associated with each digit in a numeral. Students may analyze various numbers by use of the chart. Numerous variations of this type of study of number structure may be used.

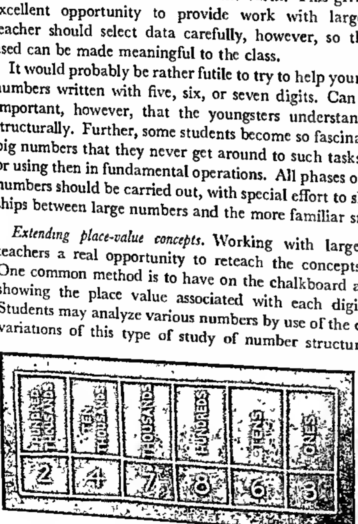

Figure 7.1

Work of this kind can be made more meaningful if the numbers used are based upon real experiences. Students may use the local newspaper to find examples of everyday use of number.

Other types of place-value activities can be used at fourth-grade level. For example, what is the largest number that can be made using the digits 1, 2, 4, and 8? Obviously, a student who is clear on place value will immediately write 8421. Does he know why this is the largest number these digits can produce? Conversely, does he know why the smallest number he can write using these figures is 1248?

Some texts introduce the comma in numerals such as 1,000 at this grade level. The comma serves only to make it easier to read the number. There is no mathematical reason for using the comma in numerals.

Ordinal numbers. As was mentioned earlier, cardinal (how many) numbers are easier for students to comprehend than are ordinal (what position) numbers. Both concepts are expanded as students progress through the grades. Some programs include work on ordinals, based upon pictorial materials, early in the fourth grade.

One of the most meaningful ways of teaching ordinals, however, is to use the students themselves as "concrete" illustrations in various ways. By the time they finish fourth grade, students should be able to describe in ordinal terms (eighteenth, ninth, etc.) any position up to the thirtieth.

Zero. The use of zero as a place holder has become familiar to most students by the time they reach fourth grade. Because of the difficulty experienced by younger children when called upon to deal with abstractions and because it is difficult to illustrate zero concretely, the study of this number is usually deferred until about the fourth grade.

It is usually emphasized that the place-holder function of zero is essentially that of keeping the other numerals in their proper positions with respect to each other. But, zero can stand by itself, completely removed from other numbers, and can convey intelligence.

For example, if you arrived late at a baseball game, a glance at the scoreboard might show that certain innings had been played but that no score had occurred. Zero can convey both items of information. A blank would indicate no score, but there might be a question as to how much of the game had been played.

It is not unusual to hear of or experience weather where the temperature is zero or below. Here, of course, the function of the zero is

to serve as a point of reference, or to mark the beginning position on a scale. Our common temperature scales use an arbitrarily defined zero, but it still is adequate as a basis for comparing temperature. Other types of scales, such as some of the instruments used in measuring electrical quantities, use the zero as a reference point. Mathematicians have other uses for the number zero. For example, it can serve as an exponent ($10^0 = 1$).

Although it is difficult to treat zero concretely, many students "see" it best in conjunction with subtraction. That is, if we take 7 apples away from 7 apples, there will be zero apples left. The absence of objects and zero can be easily related by most children.

Roman numerals. It seems necessary that we occasionally, however briefly, take another look at Roman numerals. Although Roman numerals have little social utility, occasional contact with this system should give us a renewed appreciation of our Hindu-Arabic numerals and how they facilitate number operations.

Some fourth-grade programs go as high as a hundred (C); more commonly, they go to 50 (L). The following topics usually receive some attention:

1. All numbers up to, and including, 50, when written with Roman numerals, make use of four or fewer symbols. The four are I, V, X, L. In our system, of course, all ten symbols are required.
2. Repeating a Roman numeral repeats its value. Thus, the I's in three are repeated, indicating a cumulative or additive process. This is in sharp contrast with our 111, where each place to the left increases by a factor of 10.
3. Certain letters, specifically L and V, do not repeat because VV is replaced by a new symbol, X, and LL by C.
4. If two letters are written together, with the one of greater value written first, the values of the letters are added, as in VI or XII. This is a rudimentary use of place value.
5. If two letters are written together, with the one of lesser value first, this value is subtracted from that of the other letter, as in IV or IX. For some reason, the letter V is never written in the subtractive position.

We may note that any whole number may be written with Roman numerals, but numher operations using Roman numerals become very

difficult. One theory is that in earlier centuries, the abacus was used for computation, with the results being written with Roman numerals. Thus, there was no urgent need for a zero, since an empty rod on the abacus served as a place holder.

Even in writing numbers, the Roman system is somewhat unreasonable. For instance, four symbols are required to write 8 (VIII), whereas 1000 can be written with a single symbol (M).

ADDITION AND SUBTRACTION OF WHOLE NUMBERS

By the end of the third grade, the students have studied all of the basic facts in addition and subtraction along with bridging, carrying, and borrowing. A person who viewed the situation logically, as opposed to psychologically, might conclude that no further work on these topics was necessary. Those who know children, however, and know how easily they forget, recognize that reteaching is a constant and essential task. Hence, most fourth-grade programs begin with very simple concepts of addition and subtraction, even reverting to manipulative devices if need be. Of course, the hope is that such materials may be discarded in a short time.

Addition. A common procedure is beginning fourth-grade work in arithmetic with a simple diagnostic test on the 81 facts. Such a test usually points to the need for added work on certain facts, and this should be provided. The next step might be additional work on simple column addition. It may be well for the teacher to inquire how this has been taught to her class; that is, add down or add up. The teacher may need to decide whether this topic should be retaught.

Bridging is usually retaught early in the year. More work in carrying should come in the first few weeks. Such topics as how to check a sum and how to estimate answers are periodically considered. The use of addition in problem solving gets considerable attention.

The single item that gets more attention than any other in fourth-grade study of addition is *practice work*, another term for drill. With the amount of reteaching of fundamentals that comes in fourth grade, understanding should be reasonably well developed. Hence, it is time to make the operation of addition automatic with students. Apparently, this is best achieved through extensive practice. The indexes of various texts indicate that most writers believe practice to be important. Many texts list as many as 30 references to practice exercises; some have 75

or even 100 references to such exercises. The hope is that, by now, a student can see $\begin{array}{r} 4 \\ +3 \\ \hline \end{array}$ and think 7 with a minimum of effort.

Many teachers give special attention to the correction of study habits at fourth-grade level. Students are observed closely, for example, to see whether they are using finger-counting, toe-counting, teeth-counting, or other such methods. If there are students who find such practices necessary, additional work on the addition facts will usually give them enough self-confidence so that they can discontinue these more rudimentary aids to addition.

Some attention should be given to methods used by students in column addition. Even students who have achieved mastery of the addition facts frequently use questionable methods in column addition. One practice is to listen to a student as he adds. This is the "talking out" procedure described earlier. Ideally, we would like each student to move his eyes down the column of figures

$$\begin{array}{r} 2 \\ 3 \\ 7 \\ +4 \\ \hline \end{array}$$

and think "5–12–16," not "2 and 3 are 5; 5 and 7 are 12; 12 and 4 are 16." The goal of having all your students use this technique may never be attained; but the attempt should be made.

Studies over the years have tended to verify the observations of classroom teachers about the types of errors in addition that recur most frequently in the work of fourth-graders. The most frequent error is caused by failure of students to master the 81 addition combinations. The next most frequently found errors are centered around the process of carrying.

Most fourth-grade programs give considerable attention to column addition of three-place numbers. Others go into adding four-place numbers. For the reasons explained earlier, these exercises usually come during the latter part of the year.

Subtraction. Generally, the study of subtraction in fourth grade parallels that of addition. After reteaching the 81 combinations and the principles of borrowing (usually by the decomposition method), the class concentrates on an extensive array of drill exercises in order to gain facility in this operation. The most frequent source of difficulty in subtraction is failure to know certain of the basic subtraction facts.

Probably the next most frequent source of error is in the process of borrowing. A great deal of attention should be given to these two aspects of subtraction. Again, if concrete materials seem to offer assistance to certain students such materials should be used.

There are two general directions of expansion of the subtraction concept in fourth grade. One is the application of this process in exercises containing larger numbers. Three-place, and in some cases four-place, numbers are used. Along with these, students begin to deal with complex borrowing situations, such as $\begin{array}{r} 1000 \\ -483 \\ \hline \end{array}$.

Another expansion is in the type of problems used. In the early grades, the "take away" idea was the only one employed. Indeed, upon first contact, the students probably said "take away" instead of "subtract," this being the simplest concept of subtraction. Through the grades, the subtraction concept is expanded to include several other subtraction situations. One is the "how much is gone" type. This deals with such problems as: "When Mary bought two apples, she gave the clerk a quarter. If he returned 15 cents, how much did the apples cost?" Another is the "how much more" type, such as: "If a toy costs 45¢ and Tommy has 30¢, how much more does he need in order to buy the toy?"

Still another type of subtraction is comparison, in which the student is called upon to tell how much more (or less) one number is than another. Finally, there is the "other number" type. In this, a total is given, along with one of its parts. The student is to find the other part.

It would serve no purpose to have fourth-graders learn these types or learn to classify problems according to type. The teacher, however, should be sure that students have contact with a variety of these usages of subtraction. Several such types are used in most fourth-grade programs.

Role of concrete materials. Life would be much simpler for arithmetic teachers if we could use concrete materials up to a certain grade level and then stop. Our knowledge about how people learn, however, indicates that such an approach to the use of concrete materials would not be practical. We use concrete materials whenever, according to the judgment of the teacher, such materials will help the students in effective learning. Consequently, the only suitable approach is leading students toward the abstract as rapidly as possible. Teachers above the fourth grade still find situations where concrete materials will

contribute to learning. Certainly, they should use such materials whenever there is need for them.

Teaching aids. The chief manipulative materials for addition and subtraction are counters and place-value pockets. Since these have been described in previous chapters, we do not discuss them here.

As for films and filmstrips, there are several that would be useful. Among the films are

1. *Borrowing in Subtraction,* Teaching Film Custodians, Inc.
2. *Meaning of Plus and Minus,* Encyclopaedia Britannica
3. *Subtraction is Easy,* Coronet

Other helpful filmstrips have been previously listed. There would be a special place at fourth grade for the 4-filmstrip set on "Subtraction Combinations" in the Speed-O-Strip Series, along with the companion set on addition combinations. These are sold by the Society for Visual Education. These filmstrips are designed to help students develop facility with the basic facts in addition and subtraction. The development of such facility is a major goal in fourth grade.

The teachers' manual or teachers' edition of the arithmetic text is usually a very good reference. Since the suggestions in the teachers' edition are tailored specifically to the text material, they are easy to apply. Certainly, a teacher should make full use of this usually valuable aid.

MULTIPLICATION AND DIVISION OF WHOLE NUMBERS

These two operations receive a great deal of attention in fourth grade. Indeed, some programs devote up to 40 per cent of the total page space to these topics. As you will recall, however, these were introduced to the students about a year later than were addition and subtraction, hence, we should expect the patterns of emphasis to be different.

Multiplication. Most fourth-grade programs reintroduce multiplication by going back to the basic facts. The process is related to addition by showing that multiplication is a short form of addition which is applicable only when equal groups are being combined. The terminology used in multiplication is reviewed (such as $\begin{array}{r}4\\ \times 3\\ \hline\end{array}$ being described

as "3 fours are 12"), and the students apply the process to a wide variety of problem situations.

Teachers have found numerous devices useful in helping students to see meaning in multiplication. Such materials may include trading stamps or Christmas seals, to show grouping. For example, a section of stamps may be used (see Fig. 7.2). As we view the section in this position, we see three rows of fours, or we see 3 fours. If we turn this section on end, we see 4 threes. Flash cards showing different patterns of spots are also effective in developing an understanding of multiplication. Since the basic meanings in multiplication were presented in an earlier grade, a fairly rapid reteaching is all that is usually needed in fourth grade.

Some of the 81 multiplication facts have been taught in third grade, the number varying somewhat among the various texts. One book lists 36 facts in the third-grade program. Others go somewhat higher. In general, however, the fourth-grade teacher can safely assume that about half of these facts must be retaught, and the remaining half must be taught for the first time.

Several different patterns of organization are used in presenting the multiplication facts. One practice, for example, is presenting families of multiplication and division facts, such as $\begin{array}{r}3\\ \times 4\\ \hline\end{array}$, $\begin{array}{r}4\\ \times 3\\ \hline\end{array}$, $3\overline{)12}$, and $4\overline{)12}$.

Another practice is studying all the facts with 2 as the multiplier, then doing the same for 3 and others through 9. Presumably, you will use the pattern that appears in your textbook.

A key goal regarding multiplication in fourth grade is mastery of all the facts. This, of course, requires that extensive drill be used, but drill should not be emphasized until after students have had every opportunity to develop an understanding of the process.

Several special cases of multiplication are introduced in grades

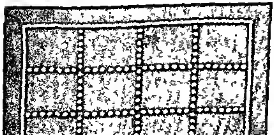

Figure 7.2

three and four. One of these is the two-place multiplicand with a one-digit multiplier. First examples should not involve carrying, as in $\begin{array}{r}14\\ \times 2\\ \hline\end{array}$. An approach that is sometimes used here is to regroup the 14 into $10 + 4$, showing each part multiplied by 2 to yield $20 + 8$. After the student understands what he is doing, this type of example offers little difficulty. Subsequently, examples with carrying are presented. The same approach is helpful. In $\begin{array}{r}18\\ \times 2\\ \hline\end{array}$, the student sees $2 \times (10 + 8)$ as $20 + 16$, or 36. With understanding, he can shorten this process to the usual procedure. Incidentally, an error that creeps in here rather easily is based upon the carry number. A student might well add the carried 1 before, rather than after, multiplying by 2, thereby getting a product of 46. If this happens, the student needs further help on the meaning of the operation. Should he write the carry number?

$$\begin{array}{r}1\\ 18\\ \times 2\\ \hline 36\end{array}$$

One teacher remarked in answer to this question, "Why give them a crutch before they are crippled?" One might well follow this principle: if the student can carry out the operation successfully without writing the carry number, he should not write it. And, incidentally, it would be good practice to go back occasionally to show that $\begin{array}{r}18\\ \times 2\\ \hline\end{array}$ is the same as $\begin{array}{r}18\\ +18\\ \hline\end{array}$, thereby reviewing the relationship between addition and multiplication.

After a student understands multiplication with two-place multiplicands, it is an easy transition to three-place or even four-place multiplicands. Two-place multipliers, however, although sometimes introduced near the end of fourth grade, get little emphasis at this grade level.

Division. Much of what has been said about multiplication is applicable to division. The two operations are usually kept "in step" with each other during third and fourth grades. For example, a program that presents 38 multiplication facts in third grade will usually include the corresponding 38 division facts at that level. The

remaining facts with these two operations are frequently presented by families in fourth grade.

Again a considerable amount of drill is necessary in order to give students confidence in using the division facts. Your text or teachers' manual may suggest ways of varying drill so as to add interest. The use of games or play activities has much less appeal to fourth-graders than is the case with smaller children. The important point is achieving mastery of the basic facts.

In the first four years of school, we ask children to learn 324 basic facts in arithmetic (zero facts being in a special category). We hope that the facts will become so familiar to the students that when they see $\begin{array}{r}5\\ \times 4\\ \hline\end{array}$ they will think 20, or when they see $4\overline{)20}$ they will think 5. Children frequently resent the degree of exactness that is required of them. When we consider that many other types of work (problem solving, for example) are going forward at the same time, we can readily see why arithmetic becomes difficult for some students.

Division is usually introduced as measurement. This is the case in which a group is to be broken into smaller groups of a predetermined size. For example, how many basketball teams (of 5 each) could be formed with 20 boys? The partition concept of division receives some attention at fourth-grade level. This is the case in which a group is to be broken into a specified number of smaller equal groups whose size is to be determined. For example, if we divide 20 boys into 4 equal groups, what would be the size of each group? The terms *measurement* and *partition* are not usually given to students. It is important, however, that they have contact with both concepts of division.

Some teaching or reteaching dealing with two-place or even three-place dividends is usually done in fourth grade. A simple example would be $2\overline{)28}^{\,14}$. This involves nothing that is new, but some students would probably want to verify their results concretely, or by regrouping 28 into its equivalent, $2\overline{)20+8}^{\,10+4}$. A slightly more difficult example would be $4\overline{)124}^{\,31}$. Such an example should not be difficult for most students, since it involves no concepts that are basically new.

The teacher should frequently point out that division may be thought of as repeated subtraction. At intervals, the concept of division

as an un-grouping process should be reviewed. This is a process by which a large group is reorganized into smaller equal groups. Also, the "guzinta" terminology (4 guzinta 8) should be carefully avoided.

A major innovation in many fourth-grade programs is the use of the remainder concept in division. Probably the best approach is to use concrete materials. When 14 is divided by 3 ("How many threes in 14?"), it will be immediately apparent that after 4 threes have been removed, there are 2 remaining. Hence, the word *remainder* is a very logical one. Although little time would be spent on concrete objects at this point, there is considerable merit in their use for a few days.

There are numerous special cases in division to which attention is usually given in fourth grade. Among these are zero in dividend and quotient, three-place dividends and quotients, and in some programs, four-place dividends and quotients. Most programs, however, limit themselves to one-place divisors. The method of checking the division operation by "multiplying back" is used extensively in fourth grade.

Teaching aids. The concrete materials that might contribute to learning in multiplication and division have been described earlier. For example, carrying in multiplication might be clearer to some students if the teacher would go back briefly to place-value pockets. The basic principle is, of course, that we use anything that is available which might clarify a process.

Several films and filmstrips that could be used have already been listed. Incidentally, why shouldn't they be used more than once as long as they are of help to the students?

Coronet Instructional Films' *Division Is Easy* and *Multiplication Is Easy* would be quite helpful in introducing or reviewing concepts. Some filmstrips that would be applicable, at least in part, are

1. "Multiplication and Division," by Young America Films.
2. The 4-filmstrip set on "Multiplication Combinations" and the corresponding set on "Division Combinations." These are in the Speed-O-Strip Series, by the Society for Visual Education.

COMMON FRACTIONS

The fractional concept is given some attention even at first-grade level, but fractions do not become a major topic of study until fourth grade. Some fourth-grade texts in arithmetic devote 5 to 10 per cent of their page space to work on fractions.

Unit and multiple fractions. Almost all the study of fractions in the lower elementary grades is confined to unit fractions, that is, fractions with 1 as the numerator. Of course, this is retaught early in fourth grade. A wide variety of teaching materials, many of which were described earlier, is available. Folded sheets of paper, fruit that has been cut into sections, the flannel board with fraction cutouts, and the fraction board are all very effective in helping the students to visualize the simple fractions.

Another expansion of the fraction concept that usually occurs in fourth grade is work with fractional parts of a group, in contrast to a fractional part of a single object. This is somewhat more difficult than is the single-object phase. Usually, fractional parts of groups are introduced concretely or semiconcretely; that is, concepts are developed by using groups of children, books, leaves, balls, or by using charts and drawings.

In the introductory phases of the teaching of fractional parts of groups, unit fractions usually get the major emphasis. At this point, however, there is a good opportunity to study equivalent fractions. Suppose we consider the pattern in Fig. 7.3. Obviously, each marble is $\frac{1}{12}$ of the total group of marbles. The top row of 3 marbles is, by count, $\frac{1}{4}$ of the total group. Also, departing from the unit fractions, the top row is $\frac{3}{12}$ of the group of marbles. Hence, the student can demonstrate that $\frac{3}{12}$ equals $\frac{1}{4}$.

Further work in the equivalency of certain fractions can be based upon some variation of the fraction board. A system is easily improvised by cutting 6 strips of heavy cardboard measuring 1 inch by 12

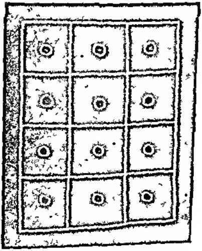

Figure 7.3

inches. One strip is left whole and marked 1. Other strips are cut as follows: 1 into two 6-inch halves; 1 into three 4-inch thirds; 1 into four 3-inch fourths; 1 into six 2-inch sixths, and 1 into twelve 1-inch twelfths. With these, it is easy to show, by matching lengths, that $\frac{2}{6} = \frac{1}{3}$, that $\frac{3}{12} = \frac{1}{4}$, and other equivalent fractions.

The study of multiple fractions usually gets major attention near the end of fourth grade. Most programs try to make the transition meaningful by showing that 2 fourths is a concept similar to 2 books, or 2 apples. Having worked extensively with unit fractions, such as $\frac{1}{4}$, the students should easily see that two of these would be $\frac{2}{4}$, and three of them would be $\frac{3}{4}$.

Throughout the study of fractions in fourth grade, students should be given opportunities to look for patterns. For example, they should be led to such generalizations as "the larger the denominator, the smaller the fraction," when a unit fraction is under consideration. Now that they have been introduced to multiple fractions, students should see that larger numerators with the same denominators describe larger fractions. Various fractional cutouts can be used to demonstrate this.

Addition and subtraction. Near the end of fourth grade, many programs introduce students to the operations of adding and subtracting fractions. Most programs, however, limit study to fractions of like denominations, so that such concepts as the common denominator are not included.

Actually, these operations may be introduced so skillfully that many students do not regard them as something new. It is easy to see that if 1 book + 1 book = 2 books, then 1 fourth + 1 fourth = 2 fourths. Likewise, if 3 dollars − 1 dollar = 2 dollars, then 3 fourths − 1 fourth = 2 fourths. Many programs do not go beyond this level of difficulty in fourth grade, and these exercises are almost universally based upon pictorial materials, rather than upon fractional symbols alone.

Mixed numbers. Most of the fourth-grade programs introduce mixed numbers but go very little beyond a mere introduction. A most common approach is by the use of measurement. For example, if a pint is $\frac{1}{2}$ of a quart, then two pints would be $\frac{2}{2}$, or 1 quart, and 3 pints would be $1\frac{1}{2}$ quarts. There are numerous meaningful applications of mixed numbers, such as $2\frac{1}{2}$ bushels, $1\frac{1}{4}$ dollars, $1\frac{1}{2}$ hours, and many others. At this grade level, just enough study of mixed numbers is included to demonstrate to the student that some quantities can be

described with whole numbers, others require the use of fractions, and still others require both whole numbers and fractions.

Aids available. Besides the aids that the teacher can readily prepare, numerous aids are available commercially. For example, Creative Playthings (New York 3, New York) has sets of "Magnefractions" that would be useful. The "pies" are available in all commonly used fractions to twelfths. Embedded magnets cause the parts to adhere to a steel-backed board. Milton Bradley (Springfield, Massachusetts) has a "Fractions are Fun" game and sets of fraction disks that are very versatile.

Of the several films that are available, one of the best is Coronet's *We Discover Fractions.* Some applicable filmstrips are

1. "Meaning of Fractions" by Young America Films
2. "What is a Fraction?" by Filmstrip House
3. "The Meaning of Fractions" by the Society for Visual Education.

MEASUREMENT

Students who have trouble with basic number operations frequently express a hearty dislike for the study of measures, since this work has certain elements that look like double jeopardy to them. One student pointed out that he obtained the right answer, but his work was counted wrong because he "didn't give the answer the right name." If an answer should be 4 gallons, it should be expressed that way, not simply as 4. Measurement is meaningless unless students can see what they are working with. To compensate in part for this difficulty, however, much of the study of measurement in fourth grade deals with measures that are of interest to the students.

Measures of length which have been studied at earlier grade levels are usually retaught. The list of units used is expanded and more accurate measuring techniques are encouraged. Since small children can get widely variant results with as simple an instrument as a ruler or yardstick, techniques in using these measuring devices should be given some attention.

Measures of units of length that are taught or retaught in fourth grade frequently include (not necessarily in order) half inch, inch, foot, yard, and mile. Except for the mile, most of these units are taught in connection with class activities. The mile can frequently take on meaning if used to measure such distances as that from home to school.

And it would be well for the teacher to be prepared to give distances from the school to various local landmarks.

Liquid measures are often used in working with recipes. Consequently, some pages of arithmetic texts look very much like pages from a cookbook. Some of the measures used in fourth grade are cup, teaspoon, tablespoon, pint, quart, and gallon. Sometimes the half pint is used interchangeably with the cup.

Dry measures, such as are used in measuring grain or fruit, are somewhat more difficult to present in the classroom. Further, dry measures do not mean a great deal to city dwellers. Hence, the list of such measures for study by fourth-graders is frequently limited to the pint, quart, peck, and bushel. Some programs point out that a bushel of apples would have a different weight from a bushel of potatoes or a bushel of wheat, thereby relating measures of weight and measures of volume.

Weight measures are usually limited at this grade level to ounces, pounds, and tons. No attention is given to a fact once prominent in arithmetic: there is more than one type of ounce (or ton). Some elementary schools record the student's weight and height on his report card. Determining and recording these measures may be used as learning experiences for fourth-graders.

Temperature measures are about the same as those studied in third grade. Ordinarily, the student works only with the familiar Fahrenheit scale. Considerable attention is given to above-zero and below-zero readings. Many activities are suggested, such as keeping a record of daily temperature readings for a week or longer.

Time measures receive major attention in fourth grade. Some frequently studied topics are seconds, minutes, hours, days, weeks, months, and years. Such concepts as A.M., P.M., and leap year are sometimes included. Telling time, as well as recording it in proper form (7:35 A.M. or P.M.), is given emphasis. Extensive use may be made of simulated clock dials or discarded alarm clocks. Further, some fourth-grade students have their own watches and may use them in class work.

Our knowledge of the learning process has caused us to present arithmetic topics on a recurring basis. It may seem logical to take up the study of time, for example, and stay on it until the topic is completed. In actual practice, however, we know that it is more effective to give students repeated short contacts with the material to be covered. Some arithmetic texts have exercises dealing with time in fifteen or even twenty different places.

Probably no phase of arithmetic offers a wider variety of opportunities for problem work than does measurement. Many good problems are presented in textbooks. Frequently, the teacher can provide even more effective problems. And why not let the students develop problems? Maybe we as teachers are guilty of confining students very closely to problem solving when they should be combining this type of work with problem recognition and development. Gratifying results sometimes may come from having students make up problems.

During the latter part of fourth grade, some attention is given to conversion from one unit to another, such as ounces to pounds, pounds to tons; seconds to minutes, minutes to hours. It is expected that students will be able to remember some of the conversion operations through using them. It would be of doubtful value to have fourth-grade students learn long lists of conversion factors, as was required of students a few decades ago.

Teaching aids. There is an abundant supply of teaching aids for measurement. With a few exceptions, the most needed measuring instruments or devices are readily available in the school or community. Others may be made by the teacher and class.

Two films that would be helpful, both by Coronet, are *Measurement* and *Story of Weights and Measures.* Young America Films series, *History of Measures,* would be applicable at fourth grade.

Some filmstrips are

1. "Advancing in Linear Measurement";
2. "Advancing in Quantity Measurement," both by Society for Visual Education.
3. "Measurement";
4. "Telling Time," both by Young America Films.
5. "How to Tell Time" (2 filmstrips) and "The Calendar" (6 films);
6. "Units of Measurement," both sets by Popular Science.

C. Using Arithmetic in Grade Four

Although there is still considerable emphasis on number and number usage in fourth grade, textbook writers usually begin introducing some applications at this level. Many problem situations are based upon games and related activities, but some attention is given to more adult matters.

MONEY

Many fourth-grade students have an allowance or have duties at home for which they are paid. Consequently, many fourth-grade teachers provide opportunities for students to study earnings, costs, savings, and other aspects of money usage. Before problems with money can mean very much, however, attention must be given to certain symbols.

The student probably has had contact with the signs for dollars (\$) and cents (¢). Nevertheless, the fact that \$2 and 42¢ is written \$2.42 can be confusing, especially since decimal fractions have not yet been studied. The usual approach is to explain simply that the point (.) separates the dollars and cents, without going into technicalities. Students should learn to visualize \$2.42 as two dollars, four dimes, and two cents. They may want to use "pennies" for "cents." It should be pointed out that "cents" is the correct word. Incidentally, at this level, the point in \$2.42 is read "and", that is, "two dollars *and* forty-two cents."

Problems involving the four fundamental operations and money are included in fourth grade. In addition and subtraction, students may be enjoined to "keep the points straight." This rule, however, does not explain why. It has been emphasized earlier that we add or subtract like quantities only. Thus, when we add

$$\begin{array}{r} \$2.42 \\ 1.33 \\ \hline \$3.75 \end{array}$$

keeping the points straight is a device to assure that we add cents to cents, dimes to dimes, and dollars to dollars. The meaning of carrying and borrowing in such operations is easy to demonstrate.

What would be your answer if a student should ask, "What would you get if you multiplied dollars by dollars?" It would be a poor pun to answer, "square dollars." Of course, such an operation with money would be meaningless. All that one could visualize here would be a certain number of stacks each of which contained a certain number of dollars, such as 3 groups of \$8 each. This would be

$$\begin{array}{r} \$8 \\ \times 3 \\ \hline \$24 \end{array}$$

Note that the 3 is *not* dollars; rather, it tells how many groups of dollars.

Confusion can ensue, however, because this principle cannot always be applied in the literal sense to division. If we use the measurement concept of division, we actually divide dollars by dollars. For example, how many groups, each containing $8, could we make from $24? Here, we would divide $24 by $8 to get the 3 groups. On the other hand, if we want to divide $24 into 3 equal groups, we would find the size of the groups by dividing $24 by 3 (not $3) to arrive at the fact that each group would contain $8. Real-life situations are perhaps best in developing problems with money.

TABLES AND GRAPHS

The presentation of data in the form of simple graphs may begin in fourth grade. Usually this consists of rather elementary pictorial graphs with a few simple questions for students to answer. Illustrative of this is the material in the fourth-grade text published by the John C. Winston Company.[1] In this text, a graph is used to compare the heights of three waterfalls. A few questions are asked about them; then it is suggested that the student look up the heights of two other waterfalls in order to make certain comparisons.

Many types of data are presented in simple tables in some of the fourth-grade books. Activities of the students may involve using these data in problems, completing certain parts of the tables, or making other tables of their own. One use of tables is the reverse of that found in previous years. Whereas earlier texts presented multiplication tables for students to memorize, many programs now give students the task of making their own tables. Through use of dittoed sheets, the teacher can frequently give the students some very meaningful exercises of this type.

D. Problem Solving

One of the major goals of arithmetic teachers is to develop skill in problem solving, and this is true throughout the elementary grades. Problem solving, however, is first given major emphasis in the middle elementary grades. This can be shown by analyzing material found in arithmetic texts. Many major concepts are now introduced through problem situations. More page space is devoted to problems than to exercises. Yet, despite all the time and

[1] Leo J. Brueckner, Edna L. Morton, and Fosler E. Grossnickle, *The New Learning Numbers* (New York Holt, Rinehart & Winston, Inc , 1956), p 287.

effort devoted to teaching problem solving, this phase of arithmetic still gives a great deal of trouble.

QUANTITATIVE THINKING

As has been mentioned earlier, some students find it difficult to think quantitatively. Such students would be happy to settle for semi-quantitative descriptions, such as large or small, and depend upon others to go beyond this point. Hence, one task of the teacher is convincing students that this unique type of thinking is essential. The best approach seems to be bringing the students into repeated contact with real problem situations which require a quantitative approach. This is far more effective in helping students to see the need for problem-solving skills than are threats or pep talks.

ABILITIES NEEDED IN PROBLEM SOLVING

One strange feature of problem solving is that this ability does not seem to follow any pattern. Some students of outstanding ability have great trouble with it; others of lesser ability seem to take it in their stride.

Since problem solving seems to require a high level of thinking, however, intelligence certainly is of importance. It is generally accepted that there are many factors in intelligence, one of which is quantitative in nature. Thus, it would not be impossible (indeed, it is a fairly common occurrence) for a person who has a high level of ability in the verbal factors of intelligence to rank considerably lower on non-verbal factors. Problem solving requires that a student have not only intelligence, but a particular kind of intelligence. What happens if a student is lacking in this respect? We still attempt to bring him to the highest possible level of achievement through skillful, patient teaching.

Another requirement is skill in reading. This, too, becomes confusing in that many students who are considered good readers have difficulty with problems. This situation arises because there are many types of reading, most of which do not require that a course of action be based upon what is read. Problem solving requires careful, analytical reading. Then a decision as to what should be done is made. Incidentally, some students become adept at finding cue words to help them make the decisions ("*at* means to multiply"). Certainly, such practices should be discouraged, since they defeat the real purpose of problem solving and frequently steer students into errors.

Usually, these sections can be purchased separately if the teacher desires. Some other well-known tests are

1. "Buckingham Scales for Problems in Arithmetic," Public School Publishing Company
2. "Los Angeles Diagnostic Tests: Reasoning in Arithmetic," Public School Publishing Company
3. "National Achievement Tests: Arithmetic Reasoning," Acorn Publishing Company
4. "Public School Achievement Test in Arithmetic Reasoning," Public School Publishing Company.

It is still true, however, that the teacher-made test is the most valuable kind for recognizing and correcting difficulties. If administered by an observant teacher, it may be used to spot error patterns in work habits as well as errors in final answers.

TEACHING AIDS

Since there is still extensive use of concrete material at fourth-grade level, the teacher is likely to use materials from the school and community in some of her problem work. For example, problems in measurement would doubtless involve a variety of measuring devices.

Because of the nature of problem solving, there is not a very extensive list of audiovisual aids available. Nevertheless, Coronet has an excellent film, *Arithmetic: Understanding the Problem.* Young America Films has a good filmstrip, "Solving Problems."

E. The Exceptional Learner

As students progress through the sequence of grades, differences in rate of learning become more pronounced. Hence, it is vital for the fourth-grade teacher to recognize and be willing to work with students at all levels of ability.

THE SLOW LEARNER

Several principles for dealing with the slow learner have been mentioned earlier. The following list presents several ideas which the teacher should remember when working with this group.

1. Since, in theory, we expect each student to work to the limit of his ability, we must accept the fact that the slow learner should stick close to the fundamentals. Whether we work with him as an individual or as a member of a small "homogeneous" group, we should strive to help him achieve a reasonable degree of proficiency in arithmetic. Only minimum attention should be devoted to "fringe" activities.
2. In fourth grade, the slow learner is likely to have considerable need for concrete materials. The teacher is confronted with the constant problem of allowing (or even encouraging) use of the concrete on occasion while, at the same time, trying to prevent undue dependence upon such aids. Any such material may ultimately retard concept development, and in this role it is a crutch.
3. The value of success as a stimulus to learning deserves great emphasis. Every effort should be made to assure the slow learner some measure of success and recognition. Some teachers find that the area of measurement offers unique opportunities here, since a slow learner could well become the best "yardstick man" in the class.
4. Very careful attention must be given to the diagnostic phase of arithmetic. Faulty work habits, erroneous understanding, failure to recall basic facts, and other such difficulties should be observed as early as possible. Frequently, correction is relatively simple when the specific difficulty has been recognized.

THE RAPID LEARNER

One danger in connection with the education of the rapid learner is that he may be content in the role of under-achiever. Hence, the teacher must be on the alert to help him achieve at a level consistent with his ability. Understanding the abstract and development of generalizations should be expected of the rapid learner.

A fast-growing list of enrichment materials is available for use in arithmetic. Some of these are *Number Stories of Long Ago*, by David Eugene Smith, available from the National Council of Teachers of Mathematics; parts of *Numbers and Numerals*, by Smith and Ginsburg; and parts of *5 Little Stories*, by Stroder. The following are available from Harper and Row, Publishers: "Games to Play," "Side Trips in Arithmetic," "The Story of Time," "Find the Number," "Crossnumber

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Puzzles," "Magic Squares," and "The Story of Zero." These paperback booklets are designed specifically for enrichment at the fourth-grade level.

A good source book is W. L. Schaaf's *Recreational Mathematics*, published by the National Council of Teachers of Mathematics. Several thousand sources of enrichment materials are cited in this book.

One point worthy of emphasis in regard to the rapid learner is that the teacher must resist the temptation to give such a student deadening tasks which are more closely associated with housekeeping than with learning. Because the rapid learner is usually willing to perform such tasks (and can spare the time for them), the teacher finds it easy to call upon him, thereby cheating him of the enrichment opportunities which are rightly his.

Something to Think About

1. What are some things you, the fourth-grade teacher, could do with a standardized achievement test in arithmetic that you could not do with a test you make yourself?
2. What are some objectives you hope to achieve in fourth-grade arithmetic that are not applicable in third grade? Fifth grade?
3. A very practical fourth-grader asks you, his teacher, why he has to study Roman numerals? How would you answer?
4. Suppose one of your fourth-graders uses a multiplication procedure that is logically unsound but yet yields good results for him. Would you require him to change to your procedure? Defend your answer.
5. A fourth-grade student in your class can handle addition very well when using the place-value pockets but has little success without them. How would you diagnose the situation?
6. Compile a list of measurement units of weight and volume as taught fifty years ago and compare it with the list of units now being taught. To what factors do you attribute the changes?
7. Suppose your fourth-grade text lists four (or five or six) steps in problem solving. Are you going to have your class learn these steps? Defend your position.

Selected References

Brueckner, Leo J., and Guy L. Bond. *The Diagnosis and Treatment of Learning Difficulties*. New York: Appleton-Century-Crofts, Inc., 1955. *Chapters 8 and 9 deal specifically with remedial work in arithmetic.*

Buckingham, Burdette R. *Elementary Arithmetic, Its Meaning and Practice.* Boston: Ginn and Company, 1953.

Emphasizes principles behind the standard practices.

Morton's book, previously cited, describes that portion of each operation that is studied in fourth grade.

Harper and Row, Publishers, has a set of eight enrichment booklets for use in fourth grade.

Spitzer's book on enrichment, cited earlier, has some helpful activities for fourth grade.

Stokes' book, cited earlier, devotes Chapter 14 to a discussion of a program for nine-year-olds.

Arithmetic in Grade Five

CHAPTER EIGHT / ARITHMETIC IN GRADE FIVE

Students should complete mastery of the basic facts for the four operations on whole numbers in grade four. *Probably no greater single stumbling block is encountered by* students in or above fifth grade than the lack of mastery of the basic facts. Hence, these must be retaught at regular intervals, with concentration upon those phases that are proving troublesome to individual students.

This chapter presents the following topics:

A. Number and Number Operations in Grade Five

B. Measurement in Grade Five

C. Using Arithmetic in Grade Five

D. Problem Solving

E. The Exceptional Learner

A. Number and Number Operations in Grade Five

A student at fifth-grade level has had several years of contact with arithmetic involving whole numbers. He has had a considerable amount of work with common fractions and may have been introduced to decimal fractions. Indeed, by the end of fifth grade, most programs have introduced all the major concepts in the traditional types of arithmetic except per cent. Considerable attention should be given to developing facility in using these arithmetic concepts.

UNDERSTANDING THE NUMBER SYSTEM

Arithmetic students of a few decades ago would be amazed at the emphasis which is now being given to building an understanding of number. This phase of arithmetic was practically non-existent then, and the chief emphasis was on memorizing and applying rules.

Large numbers. The carefully built understanding of number structure, dating back to place-value pockets in first grade, leads logically toward larger and larger numbers. Fifth-grade students are frequently called upon to work with numbers as large as 9 places, that is, to hundred millions.

Obviously, such numbers are not functional in the lives of fifth-graders. Hence, one can justify the inclusion of this type of work only to the degree that it contributes to an understanding of the number system. The students should see in large numbers a further application of principles dating back to first grade. Some types of class activities on large numbers include writing them from dictation, reading them and explaining their structure (how many millions), and using them in addition, subtraction, and multiplication.

Place value. Since few concepts are more important than place value, further attention is given to this topic in fifth grade. Not only is an understanding of place value vital in work with large numbers, it is also a phase of readiness for decimal fractions.

In some programs, students actually go back to place-value pockets early in fifth grade. Many teachers, however, find it more effective to sketch compartments on the board for use in analyzing numbers. Students work at the board or at their seats to examine number structure. For example, consider the number 247,183 shown in Fig. 8.1. Students are called upon to explain, for example, what the 7 means (7 thousands). Numerous activities may be built around such a sketch.

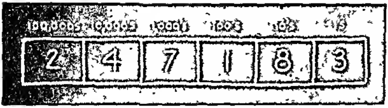

Figure 8.1

For example, what will be the effect if we interchange the 2 and 3? The 7 and 8? What is the largest number we can get using these digits? The smallest?

Some teachers like to use the place-value pockets with six or more compartments, but this may become very cumbersome. One solution would be to have color-coded sticks or cards. For example, a blue stick would represent a thousand, a yellow stick would stand for a hundred, and so forth. It is doubtful, however, whether the benefits derived from such a plan would justify the extra effort required to remember the code.

Meaning in common fractions. As students advance through the grades, dependence on concrete representations should decrease. It is difficult to present common fractions meaningfully, however, without using concrete or semiconcrete representations. There is a tendency for students to use the names "top number" and "bottom number" for the parts of a fractional numeral. The correct terminology should be used. The numeral below the line tells how many *equal* parts into which an object or group has been separated. Hence, it denominates the separation, and this numeral is called the *denominator.* The numeral above the line enumerates, or counts, the equal parts under consideration; hence, it is termed the *numerator.* Once this nomenclature is introduced, every effort should be made to establish its usage by the students.

Some programs recommend further usage of fraction boards or flannel boards with fraction cutouts at this stage of learning. It would be a good policy to use these concrete materials only with those students who actually need help in understanding the fraction concepts. Nothing is more deadening than to require work with concrete materials of those who are ready to work with abstract concepts.

Introducing decimals. Actually, students use decimal fractions when they work with money. Many programs, however, wait until fifth grade to present decimals as a separate topic. Some texts begin rather formally by telling about the decimal point as the symbol that separates

the whole number from the fraction, usually read "and." Others use a device like the odometer (the cumulative mileage indicator on an automobile) to lead into decimals. The last disk on the right moves faster than the others; hence it measures the smallest units, tenths of a mile. Other texts begin by formally terming this a new type of fraction, then citing common usages of decimals (as in the price of gasoline) to add variety.

Regardless of the approach used in presenting decimal fractions, every effort should be made to relate them to common fractions. To teach .6 and $\frac{6}{10}$ as two different things, rather than as two ways of writing the same number, would be adding needless difficulty to the task of the student.

Having presented the basic meaning of decimal fractions, many authors go immediately into the study of place value. The important feature of decimal fractions is that they represent an extension of the place-value concept in writing numbers less than 1.

One difficulty that arises early in the study of decimals is based upon the lack of symmetry in writing decimal mixed numbers. In the numeral 111.111, the 1 just to the left of the decimal point denotes "a one", whereas the one to the right of the point denotes "a tenth." Hence, the student may feel that he must "count places" two different ways from the decimal point. Actually, the numeral is symmetrical around the 1's place rather than the decimal point. If we consider the preceding numeral, the digit just to the left of the 1's place is tens, whereas the digit just to the right is tenths. For this reason, there have been proposals that we place the decimal point above the 1's place, thereby making the decimal point the central figure. These proposals, however, have not been widely accepted.

Roman numerals. Most programs review Roman numerals in fifth grade. Some of them consider an additional numeral (D for 500). It is common practice to review the basic rules for writing Roman numerals. Some practice in changing numbers written in Roman numerals to the Arabic system, and the reverse, usually concludes this phase of the fifth-grade program.

Rounding numbers. Many arithmetic programs put considerable emphasis upon the value of estimating answers before carrying out the computations in a problem or exercise. Estimating answers, however, presupposes a knowledge of rounding numbers. This concept (rounding off) is surprisingly difficult for many students. If all numbers

were rounded downward; that is, if all numbers between 300 and 400 were rounded to 300, it would be easy. If, however, the assignment is to round to the nearest hundred (the usual method), the student must decide which *is* the nearest hundred. Some teachers make this a mechanical operation by suggesting that all numbers above 350 be rounded to 400. It is important, however, that the student understand just what he is doing; why it is important; and how the process is carried out. If he is clear on the structure of number, he will not need a rule in order to round to tens, hundreds, thousands, or to any other value.

NUMBER OPERATION WITH WHOLE NUMBERS

Although students at the fifth-grade level work extensively with fractions, the chief emphasis is still on whole-number operations. It is not unusual to find fifth-grade texts in which 70 to 75 per cent of the total page space is devoted to whole-number work.

Addition and subtraction. It is never safe to assume that the addition and subtraction facts have been mastered. Therefore, early in the school year, many teachers give a mastery test, the results of which are used to indicate reteaching needs. Then diagnostic tests are frequently used in order to spot difficulties in adding or subtracting two-, three-, or four-place numbers. Carrying and borrowing receive special attention in the reteaching of these topics.

Column addition is given special emphasis in many fifth-grade classes. This does not involve anything new so far as basic principles are concerned, but the level of difficulty increases with the length of the column, since more operations have to be carried out "in the head." It is not unusual to find fifth-grade students adding as many as 6 three-place numbers during the latter part of the year. Again, the teacher must be very observant during the column addition work, since poor work habits frequently appear. Then, of course, there are certain patterns or errors to be observed, such as errors associated with particular combinations and errors in handling the carry numbers.

Another phase of addition that gets considerable attention in fifth grade is "adding by endings." Since this is very helpful in column addition, the two processes are frequently associated. The approach normally used in the teaching of adding by endings is to show the relationship between such exercises as 3 + 6, 3 + 16, and 3 + 26.

After understanding has developed, practice will help most students develop facility in this type of addition. It is doubtful strategy, however, to require students to learn as addition facts the vast array of "adding by endings" combinations.

In subtraction, fifth-graders work toward the use of larger numbers. They study the different types of situations in which subtraction should be used. Most of the time devoted to subtraction, however, is spent in working on a wide variety of practice or drill exercises to develop speed and accuracy in subtraction.

Estimating sums and differences. For many years, students have been called upon to apply the sensible answer test to the results obtained from a problem. This is still widely used and can be quite effective. Some programs are going beyond this point, however, and are having students estimate answers before they work the problem. This approach does not usually receive major emphasis in fifth grade. A student who can estimate reasonable answers before working a problem is demonstrating a commendably high level of understanding. For most students, estimating answers could be a meaningful activity, but a weaker student who finds it impossible to make reasonable estimates should probably be spared undue agitation regarding the process.

Multiplication and division. Again, diagnosis of difficulties with the basic facts usually comes early in the year. This, of course, is followed by review of those facts which appear to give trouble.

Many teachers like to review, however briefly, the fact that multiplication may be considered a short form of addition. Others go back to basic principles in order to explain some of the operations. For example, the exercise 12×24 may be written as follows:

$$\begin{array}{r} 24 \\ \times 12 \\ \hline 48 \\ 240 \\ \hline 288 \end{array}$$

Students should be reminded that in multiplying by the 1, they are really multiplying by a 10. They can then see the reason for shifting the second partial product one place to the left. The zero in 240 is usually omitted as non-functioning, yet it does show the sense of a much-mechanized operation.

Some special cases in multiplication are usually taken up in fifth grade. For example, in

$$\begin{array}{r} 24 \\ \times 30 \\ \hline 720 \end{array}$$

we simply think that any quantity multiplied by zero will yield a product of zero, and this is written as shown. In some cases, multiplication by zero is written as follows:

$$\begin{array}{r} 24 \\ \times 30 \\ \hline 00 \\ 72 \\ \hline 720 \end{array}$$

The foregoing procedure does show the parallel operation between this and other kinds of multiplication examples. This system is little used now. A few decades ago, the expression would have been written

$$\begin{array}{r} 24 \\ \times 30 \\ \hline 720 \end{array}$$

By shifting the 30 as shown, the student could simply "bring down" the zero. With our present emphasis on place value, however, we could hardly justify shifting the 3 into the one's place and the zero into an unnamed place.

Other aspects of multiplication that are given attention include a study of the circumstances in which the operation is used, and how it can be used with large numbers. Also, checking of results is considered. Three approaches to checking may be presented. One, usable in such cases as 8×18, involves adding 8 eighteens. Another procedure is for the student to switch the positions of the multiplier and multiplicand and work the example again. The third is to use division; that is, if $8 \times 18 = 144$, then $144 \div 8 = 18$.

The expansion of concepts in multiplication is closely paralleled by that of division. The terminology is reviewed, conditions under which division is used are considered, and the process is broadened to include larger numbers. In some cases, up to three-place divisors are used.

Two ways of interpreting a remainder are studied in fifth grade. Students are shown that, under some circumstances, it is sensible to

leave a remainder as has been done. That is, if 11 boys are to be divided into two basketball teams, it is more realistic to have two five-man teams with one boy left over than to say we have $5\frac{1}{2}$ boys to the team. If, on the other hand, 4 apples are to be divided among 3 boys, it would normally mean that each boy would get $1\frac{1}{3}$ apples. It is easy for adults to overlook the fact that both systems for handling remainders continue to be useful.

Teachers have long recognized that division is a major stumbling block for students. There are several reasons for this. In $12\overline{)148}$, for example, the student starts at the left, whereas in all of the other operations, he starts at the right; and this is confusing. Further, his first step is to divide 14 by 12, although there actually is no 14 in the example! For years, his teachers have assured him that there are 4 tens and 1 hundred in 148, but no 14. Also, as he carries out the operation, he sees only fragments of numbers in each step. Considering these difficulties, some programs are including division by subtraction at fourth or fifth grade. For example,

$$
\begin{array}{rr|r}
12) & 1728 & 100 \\
 & \underline{1200} & \\
 & 528 & 40 \\
 & \underline{480} & \\
 & 48 & 4 \\
 & 48 & \\
 & & \overline{144}
\end{array}
$$

In this process, the student divides the whole dividend by the divisor. Then by subsequent repetitions, he arrives at the results shown. His line of thinking might be: "I can take 100 twelves out of 1728; this leaves 528. I can take 40 twelves out of 528; this leaves 48. I can take 4 twelves out of 48. So I have taken a total of 144 twelves out of 1728." Although research is still under way regarding the merits of this system, many teachers have found it to be quite effective, especially among average or below-average students.

Of course, the usual system of checking in division is multiplying the quotient by the divisor; the product should be the dividend. An ingenious ninth-grade student, however, has pointed out that there is some duplication of effort in this process. He found that he had done the necessary multiplying in arriving at the partial products in division. Hence, he was able to conserve considerable effort by using this fact,

as shown in the accompanying illustration:

```
      144
12)1728          12
   12           144
   --           ---
   52            48
   48            48
   --            12
   48          ----
   48          1728
   --
```

A glance at the index of many fifth-grade arithmetic texts will indicate that a great deal of emphasis is placed on drill exercises in division. After the basic operations are learned, development of facility with the process is a major goal.

COMMON FRACTIONS

Since the fractional concept is encountered rather infrequently in the life of a fifth-grader, it is necessary to reteach some ideas of fractions early in the year. The meaning of the denominator as the "name of equal parts" and of the numerator as "how many of such parts" is usually retaught. Concrete materials should be used if necessary.

Extending concepts. Concepts of fractional parts of objects and fractional parts of groups should be reviewed and expanded. Another meaning of fractions, that is, a fraction indicates division, is frequently introduced in fifth grade.

Many programs introduce fractions as division through social situations. For example, suppose 3 apples are to be divided equally among 5 boys. We could take the first apple, divide it into 5 equal parts, and give one part to each boy. The same could be done for the second apple, then for the third. From each apple, each boy would receive $\frac{1}{5}$. Since there were 3 apples, each boy would have 3 pieces or $\frac{3}{5}$ of an apple. Obviously, the same result could have been obtained more directly by simply thinking of $3 \div 5$. Thus, the fraction $\frac{3}{5}$ denotes a division.

Adding and subtracting unlike fractions. The processes of addition and subtraction of like fractions is reviewed in fifth grade. These processes are then expanded to include unlike fractions.

No longer does the student begin this work with a glib rule about "get a common denominator, add the numerators." Indeed, many programs steer completely away from the term *common denominator,* placing the emphasis upon rewriting the fractions so that they will be like.

One approach is the use of money, since coins have a "built-in" common denominator. Suppose we are confronted with the task of adding a quarter to a half dollar. One way, obviously unsatisfactory, is to say that 1 coin + 1 coin = 2 coins. Students object because they are not adding the same kind of coins. Many fifth-graders, however, know that, in making change, a half dollar is worth two quarters. Consequently, we could change the half dollar to two quarters, and we would then have a "common denominator" of coins. The net result is that we have reduced the two unlike fractional parts of a dollar to like quantities and, hence, can now add them to get 3 quarters, or 75¢.

There is extensive use of concrete materials in this phase of teaching. The fraction board or flannel board could help a great deal. Take the case of $\frac{1}{2} + \frac{1}{4}$. By using fraction cutouts, a student can easily conclude that $\frac{1}{2}$ is equal to $\frac{2}{4}$. Therefore, he can rewrite the problem so that he will have two like fractions, $\frac{2}{4} + \frac{1}{4}$. Throughout this part of arithmetic, emphasis should be placed on the fact that we rewrite fractions so that they are like, that is, have the same denominators. Incidentally, in the addition and subtraction of fractions, many texts list them in columns just as is done with whole numbers. This points up the fact that these operations are essentially the same for whole numbers and fractions.

If, in adding common fractions, we get an improper fraction, it may be treated as a "carrying" situation. In place-value pockets, if we got 10 ones in the ones pocket, we carried the 10 ones to 1 ten. Likewise, if we add $\frac{1}{2}$ and $\frac{3}{4}$ to get $\frac{5}{4}$, we take out $\frac{4}{4}$, or 1, carrying it to the ones place, yielding $1\frac{1}{4}$. Also, fraction cutouts can be quite effective in demonstrating this idea.

Just as carrying can be used in an addition situation, so can borrowing be used in subtraction. In $1\frac{1}{4} - \frac{3}{4}$, we "borrow" the 1, rewrite it as fourths, and thereby convert the exercise to $\frac{5}{4} - \frac{3}{4}$.

Adding and subtracting mixed numbers. Most of the principles described in the previous section apply here. Do you remember how mixed

numbers were handled in earlier decades? Take $1\frac{3}{4} + 2\frac{1}{2}$. The student would say, "4 ones are 4 and 3 are 7 to give $\frac{7}{4}$." The same approach would be used to write $2\frac{1}{2}$ as $\frac{5}{2}$. Then $\frac{7}{4} + \frac{5}{2} = \frac{7}{4} + \frac{10}{4} = \frac{17}{4} = 4\frac{1}{4}$. In more modern programs, every effort is made to make these processes like those for whole numbers. For example,

$$\begin{array}{rcr} 1\frac{3}{4} & = & 1\frac{3}{4} \\ \underline{+2\frac{1}{2}} & = & \underline{+2\frac{2}{4}} \\ & & 3\frac{5}{4} \end{array}$$

Upon carrying $\frac{4}{4}$, or 1, to the ones place, we get $4\frac{1}{4}$. Note that this process does not confront the student with any basically new concepts.

Reducing fractions to lowest terms. Several textbooks introduce the process of reducing fractions during fifth grade. Once again, $\frac{1}{4}$ and $\frac{1}{4}$ equals $\frac{2}{4}$. By the use of fraction cutouts, rubber pies, or any of the other materials of this type, however, he can readily verify that $\frac{2}{4}$ equals $\frac{1}{2}$. At this point, the teacher may ask what could be done with $\frac{2}{4}$ in order to get $\frac{1}{2}$ from it. Many students will see that this can be achieved by dividing both the numerator and denominator by 2. Ultimately, the teacher will show that

$$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

The procedure for carrying out the mechanics of the operation should be presented after the students see what is taking place, and not before. It should be established that a fraction is in lowest terms, or standard form, when both the numerator and denominator cannot be divided evenly by any number except 1.

Teaching aids. The most valuable teaching aids in common fractions seem to be some type of fraction strips or cutouts by which the students can visualize the processes with fractions. These would be especially helpful in introducing processes which require finding like fractions.

The Coronet film, *Fractions: Finding the Common Denominator*, is quite effective. Several filmstrips which would apply, at least in part, are

Filmstrip House

1. "Working with Equal Fractions and Reducing Fractions"
2. "Working with Like Fractions and Improper Fractions"
3. "Adding with Fractions"
4. "Subtracting with Fractions"

Society for Visual Education

1. "Changing the Terms of Fractions"
2. "Adding Like Fractions and Mixed Numbers"
3. "Subtracting Like Fractions and Mixed Numbers"
4. "Adding Unlike Fractions and Mixed Numbers"
5. "Subtracting Unlike Fractions and Mixed Numbers"

DECIMALS

In some programs, decimal fractions are introduced in fifth grade. The expansion of our base-10 system of numerals to a way of writing numbers less than 1 can take place so naturally that students may not realize that something new has been presented. Further, where the decimal system is understood by students there should be little difficulty with the major concepts.

Most programs emphasize the relationship between common fractions and decimal fractions. For example, $\frac{1}{2}$ dollar or 50¢, may be thought of as $\frac{50}{100}$ or .50 dollar. The fifth-grade program commonly includes the study of tenths and hundredths.

The operations of addition and subtraction with decimal fractions and mixed decimals are usually studied. Since carrying and borrowing in these operations are so similar to the same processes with whole numbers, little difficulty is ordinarily encountered. It is emphasized that the rule about "keeping the decimals straight" is a logical device to insure that we add like quantities.

Students are given some experience in reading decimal fractions aloud, writing them from dictation, and applying them to coins, odometers, and rulers calibrated in tenths of an inch.

B. Measurement in Grade Five

As students progress through the grades, their work with measures should undergo several changes. New units of measurement may be introduced in each grade. Also, the skill with which students use instruments, hence the accuracy of measurement, should improve. Further, it is hoped that they will make progress in their basic understanding of measurement. For example, a student's ability to estimate distances (a good indicator of understanding) should improve.

Extending concepts. Length is probably the first kind of measure used by children. Although most of the common units for measuring length are introduced earlier than fifth grade, they are frequently retaught. Units of length that are studied usually include inches, feet, yards, rods, and miles. Most of these units are taught through activities involving rulers, yardsticks, and perhaps a steel tape. Many measures inside and outside the classroom may be made, with the students doing some estimating and comparing with results of their measurements. Also, considerable attention is given to the relationships among the various units of length and converting from one unit to another.

Liquid measures usually included for teaching or reteaching in fifth grade are the cup (or half pint), pint, quart, and gallon. Since many school lunchrooms serve milk in half-pint cartons, they are readily available for use. In general, the same types of activities that were mentioned for measures of length may be used, such as developing concepts of size through use of measuring devices, converting from one unit of measure to another, and estimating measures. One difficulty sometimes arises from teachers' overuse of some easily available measuring devices, such as milk cartons or bottles, cups, and fruit jars. A student may come to associate the word "quart" with a particular *shape* of container. This can be avoided if the teacher is careful to use assorted shapes of containers.

Dry measures commonly included in fifth grade are the pint, quart, peck, and bushel. Along with various activities, such as those previously described, the student is usually informed that a bushel of one material may weigh considerably more, or less, than a bushel of another material. For example, a bushel of peas weighs 60 pounds, whereas a bushel of oats weighs only 32 pounds. The amount of time given to the study of dry measures is gradually being reduced, since urbanization is resulting in less student contact with such measures.

As emphasis on dry measures has decreased, attention to measures of weight has increased because many types of commodities (such as fruits and vegetables) formerly sold only by the bushel, half bushel or peck, are now sold by the pound. Further, several units of weight that were formerly taught no longer appear in arithmetic texts. Indeed, many fifth-grade texts now present only ounces, pounds, and tons. Simple scales are useful as a teaching aid. Some teachers like to have the students estimate weights, which is slightly more abstract than estimating distances.

Temperature measures are usually limited to the Fahrenheit scale. Frequently, the principles of operation of the thermometer are discussed as a "fused" science-arithmetic activity. The method of reading a thermometer is studied. Temperatures are measured inside and outside the classroom, above, and on one side of, the radiator, in boiling and freezing water. Records of outside temperature over a period of days or weeks are kept. Some programs include the clinical thermometer and the technique of measuring body temperature. Amounts of increase or decrease, differences between readings, and local temperature problems are considered. The teacher should note, however, that it is incorrect to use such terms as "twice as hot" or "half as cold" in working with Fahrenheit readings. *Do you know why?*

By the end of fifth grade, students have studied numerous units for measuring time. Among them are the second, minute, hour, day, week, month, year, and century. Practice for those who need help in reading the time should be provided. The use of the "second" hand on a watch may be included. A great deal of attention is given to such activities as converting from one unit to another. Some programs go into the meaning of A.D. and B.C. Problems involving time in everyday life are stressed. There is still some difficulty, even at fifth grade, in getting certain students to see the "reality" of time measures, since such measures do not yield themselves to concrete treatment in the way that most other kinds of measures do.

Several of the topics studied in measurement, notably time and temperature, are ideally suited for combining with other subjects. In studying certain phases of time, for example, arithmetic and geography could be correlated. The teacher should seek to show the application of arithmetic to other subjects where possible. The idea of arithmetic as something that happens only in arithmetic class can be minimized by such practices.

Square measure. Many fifth-grade programs introduce measures of area. This, of course, should be preceded by some work on simple geometric figures, such as the square, rectangle, and possibly the triangle. These are usually presented by using familiar objects, like a scarf or handkerchief, unfolded to produce a square or folded to produce a triangle; table tops; chalkboards; the floor of the classroom; and play areas. Having examined a square table top and compared it with a rectangular one (a bit of confusion can result from the fact that a square is actually a special kind of rectangle), students can frequently develop their own statements about how the two shapes of table tops differ.

This is more effective than to have them begin by learning definitions of geometric figures.

After the students have learned to recognize the geometric figures, most texts proceed to the study of perimeter. When given an opportunity to do so, many students can devise their own method for finding perimeter. For example, in the rectangle shown in Fig. 8.2, one student might think, "2 feet + 12 feet + 2 feet + 12 feet = 28 feet, so the perimeter is 28 feet." Another might use this line of thought: "There are two 2-foot sides and two 12-foot sides. So

$$\begin{array}{r} 2 \\ +2 \\ \hline 4 \end{array} \quad \text{and} \quad \begin{array}{r} 12 \\ +12 \\ \hline 24 \end{array}$$

and 24 feet + 4 feet = 28 feet." Another way of thinking would be "2 feet + 12 feet = 14 feet; since there are two of each such measures, the perimeter would be 14 feet × 2 = 28 feet." Also, there will always be the youngster who needs to take a yardstick or folding rule and measure each side. Students, however, need the opportunity of coping with unfamiliar situations occasionally, and such opportunities are abundant in the study of the perimeter.

Many teachers like to introduce the measurement of area by providing students with cardboard blocks cut into square feet and square inches. Then students may measure the area of selected surfaces by actually placing the square-foot or square-inch blocks on them. After the concept of area has been established, the blocks are discarded.

By a variety of laboratory experiences, such as that described in the previous paragraph, students are led to discover that there is an easier way to find the area of a rectangle, this being the product of length and width. Some teachers present this concept with such interest-holding devices as a pan of fudge, the culminating activity being that each member of the class receives one square inch of candy.

After the students have become acquainted with the concept of measuring area, many teachers let them work with three units: square inches, square feet, and square yards. Although the square mile will

Figure 8.2

come later, it is of doubtful value to fifth graders. Charts are available, or can be prepared in the classroom, which show graphically that a square foot contains 144 square inches, and a square yard consists of 9 square feet. Some work on converting from one set of units to another is usually given; however, this is not a point of major emphasis at fifth-grade level.

It was mentioned earlier than many denominate numbers, that is, numbers associated with names, cannot be multiplied by the same denominate numbers to yield sensible results. For example, $2 × $2 cannot make sense nor can 2 gallons × 2 gallons. Area or square measure of a rectangle may be considered the number of units of length times the number of units of width, where length and width are expressed in the same units. In teaching area, some writers prefer to make the multiplier abstract and the multiplicand a denominate number expressed in square units.

Teaching aids. Probably in no part of fifth grade arithmetic is there a more abundant supply of teaching materials than in measurement. Further, these materials are absolutely vital, since a student must have some experience with the various measurement devices for measurement to be meaningful. Also, these materials are readily available at school or in the community. Just say, "We will need a few pint and quart jars brought from home," and you will probably have more of them than you can store. The lunchroom supervisor, science or physical education teacher, and many others can help provide materials for teaching measurement.

Many of the school supply dealers have assorted materials, such as clock dials, thermometers, and rulers, available. For teaching perimeters and areas, Educational Supply and Specialty, Huntington Park, California, has a "One Square Yard" board and a "Perimeter Area Board" that would be helpful.

Certain films, some of which were mentioned earlier, could be helpful. Coronet has three films on measurement at this grade level: *Measurement*, *Measuring Areas: Squares, Rectangles*, and *Story of Weights and Measures*. Many of the filmstrips cited in the previous chapter would be useful at fifth-grade level.

C. Using Arithmetic in Grade Five

As has been noted, much of the arithmetic in earlier grades centered around games. Drill on the basic facts was frequently incorporated into play activities. By

fifth-grade level, however, many students have outgrown this type of teaching and are ready for a more straightforward approach.

Business usage. It is, of course, too early to go into "business arithmetic" as such, but many textbooks include certain simple business practices for fifth-graders. A few real-life situations used by texts are buying and paying for school lunches, keeping a record of expenses on a trip, grocery shopping, earning and spending an allowance, and working for various rates of pay. In these and many related activities, students are brought into contact with elementary but basic business practices.

Graphs. Another example of applied arithmetic in fifth grade is the use of graphs. Work on graphs has a great appeal to those students who like to think in an orderly pattern and is a great bother to those who are not particularly systematic. Line and bar graphs can, however, give students valuable training in seeing relationships between sets of data.

Many teachers prefer to depart from textbook material and make use of locally obtained data in working with graphs. For example, a graph could be used to show the scores in a game, the temperature reading each morning for a week or so, rainfall, and other kinds of data. Usually, students first become familiar with graphs by reading data from them. Then they are given an opportunity to make their own graphs, using data which they have secured.

Frequently picture graphs are introduced along with the bar and line types. These, of course, involve a slightly different concept in that students are called upon to multiply each symbol by a fixed multiplier. Usually, work with picture graphs is limited to reading, rather than preparing, such materials.

Some texts include work in reading and preparing tables. Incidentally, it is easy for adults to overlook the fact that special skills are involved in reading a table. Since tabular material is usually of the same general nature as that used in line and bar graphs, the work on tables and graphs is frequently correlated.

Scale drawings. Another applied arithmetic topic that frequently enters the fifth-grade program is that of scale drawings. This type of work can be made most meaningful by using local situations. Some teachers start with a scale drawing of the classroom. Others prefer a playground area, such as a baseball diamond. After the concept is established, several other types of drawings can be used, such as simple

house plans and road maps. Generally, fifth-graders limit their activities to reading rather than preparing scale drawings. Even at this level, however, such work can be valuable in introducing the concept of proportionality.

Teaching aids. Generally, it is accepted that you cannot do much with graphs except by working with them. Certainly, the same holds true for scale drawings. So, whether such materials be teacher-made or obtained from other sources, teaching materials are vital in this type of work.

Two of the Coronet films would, at least in part, be of assistance on these topics. They are *Maps Are Fun,* which might also be useful in geography, and *The Language of Graphs.* Several previously cited filmstrips would also be useful.

D. Problem Solving

Developing a reasonable proficiency in solving "word problems" should receive emphasis in fifth grade. Although we still seek problem situations that are real and interesting, we also move toward problems of greater complexity. A higher degree of accuracy is emphasized. And, of course, each year students learn additional number operations to use in problem solving.

Some general principles. Since one of the prime goals in problem solving is that students learn to analyze a verbal statement and decide which number operations should be used, no uniform approach or formula is usable. On the other hand, the teacher should do more than assign problems, leaving the students to "sink or swim."

Since it is vital that each student experience some success, problems of varying degrees of difficulty should be used. We should, however, recognize the fallacy of "giving Johnny the addition problem, since he can do that kind." The student should be challenged. Yet the material should not be so difficult that he is constantly defeated.

The teacher should be an active participant in the teaching of problem solving. Besides selecting problems of appropriate levels, she should give guidance when needed in analyzing problem situations. This might include a series of questions designed to lead students forward in small steps. And there may be many sessions, usually individual in nature, where the teacher may say, "Now what are we seeking?" Or, "What do we know that would help?" Or, perhaps, "Where do we go from here?" It is a skillful teacher who can participate in analysis up

to a point where the student can continue on his own and leave him to complete the task.

The teacher has a third function in problem solving, that is, looking for clues to remedial work that might be needed. As mentioned earlier, diagnosis of difficulties is primarily a matter of teacher attitude. During periods of problem solving, many students show evidences of poor work habits that they do not normally exhibit. This, then, is a good time for the teacher to be on the alert for symptoms of basic difficulties.

Some techniques. Several phases of problem solving receive special attention during fifth grade. One of these is estimating answers. It is safe to assume that students who consistently give reasonable estimates of answers before finding solutions are thinking correctly about the problems. Hence, some textbooks frequently call upon the student to estimate answers before solving problems.

Mental arithmetic, once the pride and joy of the mental discipline advocates, is included in some programs as a part of problem solving. This, you will recognize, is similar to estimating answers, except that in mental arithmetic fairly easy numbers are used so that most students can handle the operations mentally. Some teachers believe that the "no-pencil" problems appeal to their strong students but definitely do not draw enthusiastic response from average and below-average students.

The mechanical solving of problems according to a given set of steps has largely been discarded. Many teachers like to see how many different ways their students can devise for solving problems. Every class has its quota of non-conformists. It is to be hoped that teachers will view such students as challenges rather than as nuisances. No phase of arithmetic offers more opportunities in this direction than does problem solving.

Other techniques that are useful with fifth-graders in their work with problem solving are (1) using analogous but simple problems in order to comprehend a more complex process, (2) drawing diagrams to help visualize a problem, (3) noting similarities among situations; that is, a student who can solve a particular type of problem dealing with measures in feet should be able to solve a comparable type of problem using yards or miles.

Teaching aids. Since the goal of the teacher is to move from the concrete to the abstract as rapidly as possible, many teachers prefer to keep fifth-grade problem work on a fairly abstract level. If, however,

there are students who still need to use concrete or semiconcrete materials in problem solving, presumably such materials should be used.

The Coronet film, *Arithmetic: Understanding the Problem*, would be applicable. Also, parts of Coronet's *How to Find the Answer* could be used. "Solving Problems," the Young America Films filmstrip, is quite effective.

E. The Exceptional Learner

With each advance in grade level and the corresponding increase in chronological age, the ability range in a class increases. As a result, the problems associated with exceptional patterns of learning become more pronounced. Indeed, one can hardly conceive of a group, even a so-called homogeneous one, where these problems do not exist.

The slow learner. Many teachers recognize in their classes two distinct types of students with arithmetic difficulties. The first consists of those youngsters who have the ability to do the work but who, because of fear, inadequate background, or other such complications, have come to accept the idea that "I can't do arithmetic." Too often, parents try to console such students by pointing out that "He came by it honestly; I had the same trouble." There are two approaches available, singly or together, for the teacher. One is very close observation of the student, his work, and work habits, in order to recognize points of difficulty. After diagnosis, treatment is often relatively easy. Another approach is to give the student work at a lower grade level so that he can experience a high degree of success. Nothing is more effective in treating this type of difficulty than to have the student prove to himself that he *can* work arithmetic.

The second group of students with arithmetic difficulties is made up of pupils who actually lack the ability to achieve at a high level. In working with this group, the teacher should keep several items in mind:

1. The low-ability group requires a great deal of drill. There is little opportunity for them to branch out into the interesting byways of arithmetic. They normally show little initiative and require rather close supervision in their work. Certainly, every effort should be made to use drill in its more palatable forms, but there is no alternative to the use of repetitive work.
2. Having experienced failure in many endeavors, these students are frequently inclined to accept failure as their lot in life, hence, to

relax and not even try. The best remedy for this situation is to guarantee some degree of success by using simplified assignments for them.

3. It is a part of the slow learner's pattern of operation that he has special difficulty with abstract reasoning. Therefore, he will normally require much more work at the concrete and semi-concrete levels than will the students who possess average or above-average ability.

The authors, in working with individual students who are having trouble with mathematics, have observed that fifth grade is an especially important year. Children who failed to learn a process, or who learned it incorrectly, frequently get as far as fifth grade before the situation becomes critical. A few actual cases illustrate this situation.

Pupil A had difficulty with addition, but neither parents nor teacher could be more specific as to the difficulty. Work with the flash cards indicated that he had an adequate knowledge of the basic addition facts. By having him work aloud, it was noted that column addition gave him a great deal of trouble. Further testing and observation indicated that, in column addition, he almost invariably stumbled when he added a 2. He could add in 7's or 9's quite accurately, but a 2 would throw him. Concentrated drill on this corrected the difficulty in short order. But, unfortunately, this was just one of pupil A's arithmetic deficiencies.

Pupil B had been in overcrowded classrooms for her first three grades, so that there had been inadequate opportunity for individual help from the teacher. As a result, if she learned something incorrectly, it was very likely to go unnoticed. By the time pupil B was a fifth-grader, it was apparent that she was having trouble with multiplication and division. A simple oral test indicated that she was able to use the basic facts quite well. Since multiplication skills are essential in long division, she was given some work-aloud exercises in multiplication. It developed that, in multiplying 207 by 7, she was thinking "7 sevens are 49, put down the 9 and carry the 4; 7 fours are 28, put down the 8 and carry the 2." Intensive reteaching, followed by drill, cleared up this point in relatively short order.

The rapid learner. Many fifth-grade texts have suggestions, variously entitled, for enrichment activities. In other arithmetic series, these suggestions appear in the teachers' guide. Nearly all texts now provide some form of assistance to teachers in their work with rapid learners.

Many fifth-grade pupils could make profitable use of some of the reference materials that are available. For example, good material on many mathematical topics is to be found in the more popular encyclopedias.

It would be impossible to cite all the available sources of enrichment here but, the Harper and Row series is outstanding. Some of their titles (fifth-grade level) are "The Story of Measures," "Crossnumber Puzzles," "Jokes and Riddles," "Numbers Do Strange Things," and "Ways to Multiply."

Schaaf's *Recreational Mathematics*, published by the National Council of Teachers of Mathematics, should be a valuable aid to the teacher. This is actually a guide to the literature, or a "source of sources." One of the eight chapters lists sources available to the teacher for assistance in arithmetic and algebra. The best materials however, will be ineffective unless wisely used. The first requirement to be met in challenging the rapid learner is that the teacher recognize the need for enrichment. Then she must try to secure the needed materials and make them available to the rapid learner.

Something to Think About

1. As a new fifth-grade teacher, you find that most of your class is ready for fifth-grade work. Four of them, however, do not know the addition facts. How would you handle this situation?
2. Considerable attention is being given to use of the number line in the various operations. Read and summarize two or three articles dealing with this subject.
3. A great deal of research is being done in the general area of problem solving. Give the class a summary of one published report based upon research of this type.
4. Why shouldn't a student refer to the parts of a fraction (fractional numeral) as "the top number" and "the bottom number?" Aren't these as descriptive as "numerator" and "denominator?" Defend your answer.
5. A student was clear on the meaning of division as "un-grouping" in earlier grades, but he is confused about how his line of thinking would apply to $3 \div 5$ or $\frac{3}{5}$. How would you explain it to him? Use references as necessary.

6. Do you have trouble with problem solving? Make a list of difficulties you may have experienced and analyze them for patterns and possible remedies.

7. Prepare a list of problem situations that could occur in the lives of fifth-graders. Show how one of these could be used in the teaching of arithmetic.

Selected References

Devault, M. Vere. *Improving Mathematics Programs.* Columbus, Ohio: Charles E. Merrill Books, Inc., 1961.

A collection of readings on teaching arithmetic.

Hickerson, J. Allen. *Guiding Children's Arithmetic Experiences.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1952.

Emphasizes the contribution of arithmetic to the child's growth and development.

The National Council of Teachers of Mathematics has published a booklet called "The Revolution in School Mathematics" (1961) which describes some of the changes that have occurred in mathematics. This booklet gives implications for teachers at all grade levels.

Harper and Row's enrichment booklets for fifth grade.

Smith, David Eugene, and Jekuthiel Ginsburg. *Numbers and Numerals.* Washington, D.C.: The National Council of Teachers of Mathematics, 1937.

Although not specifically designed for any grade level, parts of this fascinating book would be of interest to fifth-graders.

Catalogs published by supply houses may serve as a source of ideas for improvising teaching materials.

Arithmetic in Grade Six

As students advance through the grades, the topics emphasized in arithmetic change according to grade level. In the lower grades, a major effort is made to develop understanding of numbers and our system of numeration. This continues to be a goal of arithmetic teachers at all grade levels. With each succeeding year, however, more attention is given to learning the basic facts for the four fundamental operations on whole numbers. These, in theory, are mastered by the end of fourth grade.

By the time the student enters sixth grade, he has had contact with most of the major phases of arithmetic. Hence, the chief goals in sixth-grade arithmetic are (1) the development of a higher level of understanding of the arithmetic processes which were introduced earlier; (2) the introduction of a limited number of new arithmetic processes; (3) the development of facility in the use of the arithmetic processes in a wide variety of applications.

The following topics are presented in this chapter:

A. Number and Number Operations in Grade Six

Much of the work in number operations in sixth grade is review. Most texts cover these topics rapidly, with the prime purpose of reestablishing certain skills that may have been lost through disuse.

UNDERSTANDING THE NUMBER SYSTEM

One might well wonder how long the teacher has to bother with this phase of arithmetic. Does the time ever come when a teacher can assume that her pupils understand the number system? It is the nature of learning that repeated contact with a concept is vital to understanding. Hence, even at sixth grade, some attention is given to the development of an understanding of our number system.

Large numbers. In sixth grade, the students usually work with numbers in the billions. This includes reading and writing such numbers and using them in problems. When such numbers are first introduced, many teachers like to go back for a quick review of place value in order to show that numbers with many digits do not differ structurally from numbers with a few digits.

In some texts an effort is made to show that, in writing large numbers, there is a recurring pattern of place value. For example, see Fig. 9.1. To many students, this "simplification" would actually complicate the matter. Certainly, this device should not be used unless it clarifies the structure of our system of numeration.

Some educators say that numbers in the billions fail to meet the social criterion because they do not function in the lives of sixth-grade

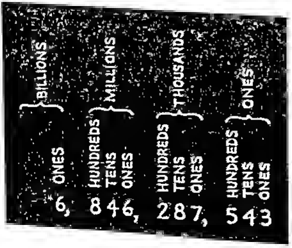

Figure 9.1

students. Although it is probably true that sixth-graders would not use such numbers in daily living, these numbers do figure frequently in the news which they read and hear. Many industries deal with such numbers in annual payroll and production figures. Space travel discussions make use of such numbers in describing distances, and there is always the national debt.

Decimals. The various textbooks differ considerably in the amount of work with decimals given at sixth-grade level. Some go to thousandths, others to ten-thousandths, and others to millionths. There is agreement, however, about the objective. A major goal is that the students have an understanding of the structure of decimal fractions. Hence, place value (frequently using place-value pockets) receives considerable emphasis. Along with this, many texts stress that decimals represent an extension of our number system, so there is no need to think of them as some kind of new number system. Our system of money is still referred to as an illustration of "decimals in action."

Rounding whole and decimal numbers. It was mentioned earlier that the student, if he is to apply the reasonable-answer test, must be able to round off numbers. Somehow, this process gives considerable trouble, even to college students. Yet it involves nothing except an understanding of place-value.

Most arithmetic programs associate rounding off with real-life situations. Further, this process is usually studied in connection with its most valuable application, that is, estimating answers.

Roman numerals. Most texts present a review of some of the basic principles of Roman numerals. The fact that there is no Roman numeral for zero is reviewed. Along with this, it is pointed out that place value is used only in a very rudimentary way and is based upon an additive or subtractive process. The weaknesses of this system of numeration for use in the fundamental processes are reexamined.

The use of Roman numerals in writing large numbers is presented. Some texts give the students practice in reading and writing numbers to a thousand. Others go as high as two thousand (MM).

It is important that the teacher keep work with Roman numerals in its proper perspective. Since this system has little value to us today, it is usually studied as being of historical interest and as a means of helping the students to develop an appreciation of our Hindu-Arabic system. Most sixth-grade textbooks give very little space (frequently only one or two pages) to the Roman system.

NUMBER OPERATIONS WITH WHOLE NUMBERS

Although whole-number operations have been studied every year since first grade, the sixth-grader still has some work to do in this area. If we assume that the student has made normal progress to this point, his work with whole numbers in sixth grade will consist largely of (1) a review of the fundamental meanings of the processes, (2) rapid review of the basic facts, (3) application of the facts to new problem situations.

Addition and subtraction. Addition for sixth-graders frequently begins with a series of diagnostic tests. This is logically followed by reteaching of the weak points as revealed by these tests.

Variations or expansions can take several directions. One is the use of longer columns. It is not unusual for sixth-graders to be called upon to add one-place numbers in columns of 8 or three-place numbers in columns of 6. Some programs go as high as five-place numbers, usually with a limit of four addends. Another variation is that of adding numbers written from left to right rather than in columns. Further attention is frequently given to estimating answers before addition since this requires the exercise of judgment at a relatively high level.

The work in subtraction offers few possibilities for variation. Hence, after students review basic principles and the subtraction facts, most of the time spent in subtraction is given to dealing with larger numbers. Minuends of 5 or 6 places are frequently used. The process of subtraction with borrowing is reviewed. Some special attention is given such cases as

$$\begin{array}{r} 1{,}000 \\ \underline{-243} \end{array}$$

where the sequence of zeros introduces a complication.

Some texts provide a considerable amount of drill work in subtraction. Others shift the emphasis to such topics as "How do you decide when you should subtract?" The latter approach certainly has much to offer, but it is of doubtful value to a student if he knows *when* to subtract but does not know *how*. Both the when and the how of subtraction are important and therefore should be studied together.

Multiplication and division. In addition to reviewing the basic facts of multiplication and division, sixth-graders give attention to a number of other phases of work with these operations. For example, many

programs include a certain amount of reteaching the fundamental meanings of these operations (multiplication as repeated addition, division as repeated subtraction). Along with this, the students consider the circumstances under which each operation would be used.

The use of these processes with larger numbers is included in sixth grade. Two- and three-place divisiors and multipliers are used, and some texts go to even larger numbers. Further attention is given to the use of denominate numbers in multiplication and division. The circumstances under which one denominate number may be multiplied by another (feet × feet to yield square feet) are reviewed. In division, the measurement and partition meanings are reviewed and problems involving each meaning are presented.

With the hope that understandings will be developed, many teachers encourage students to estimate products and quotients before working a problem. The "sensible answer" check, however, seems to be far more popular with teachers than with pupils.

Teaching aids. As students progress to higher grades, usually less attention is given to the use of teaching aids. An exception to this pattern is those topics that are introduced at the higher grade levels, such as work with common fractions. Concrete materials have much to offer in the development of understanding in these areas.

Nevertheless, it is well for the sixth-grade teacher to remember certain basic principles that apply to the use of concrete materials. Some of these are (1) Use concrete materials only when they assist in the development of understandings. (2) Move away from the concrete and toward the abstract as rapidly as possible, since there is always danger that students may come to depend on concrete materials. This, of course, would mean that a teaching aid has become a crutch. (3) Go back to the use of concrete materials, usually for brief periods, whenever there is need for a review of basic concepts. (4) Since the need for concrete materials is a highly individualized matter, we cannot expect all the members of a class to make the transition from concrete to abstract at the same time.

COMMON FRACTIONS IN GRADE SIX

Previous contact with common fractions cannot be taken to mean that students have achieved satisfactory understanding. Consequently, the first work with fractions for the sixth-grade class is usually a review of certain concepts: what the numerator tells us; what the

denominator tells us; what the two terms tell us when written together; and what different meanings may be given to fractions.

Addition and subtraction. Since the fifth-grade program usually includes the addition and subtraction of like fractions (fractions with the same denominator), these topics are retaught and expanded in sixth grade. Major attention, however, is given to the addition and subtraction of unlike fractions at sixth-grade level.

Although methods of presentation vary considerably among the texts, certain approaches are widely used. A common beginning point is the addition of two fractions in which the least common denominator is the denominator of one of the given fractions. For example, in $\frac{1}{2} + \frac{1}{4}$, the student is confronted with the task of adding fractions that are obviously unlike. As in adding pennies and dimes, however, he can perform the operation if he will "make change" so as to get them in like terms. With fraction cutouts or fraction strips, it is easy to verify that $\frac{1}{2}$ can also be written $\frac{2}{4}$. Now the two fractions are in like terms and we add $\frac{2}{4} + \frac{1}{4} = \frac{3}{4}$. The student can, if he desires, verify the accuracy of his addition with fraction cutouts. At sixth-grade level, however, the teacher would probably not emphasize the use of concrete objects except for those students who obviously needed such help.

This concept is usually expanded to include several variations. For example, the addition of three unlike fractions might be taken up. At least some of these exercises would be designed to yield sums that are improper fractions. These would be converted to mixed numbers. Another variation might be the addition of columns involving fractions and whole numbers, such as

$$\begin{array}{r} 8 \\ \frac{2}{5} \\ +\frac{3}{10} \\ \hline \end{array}$$

Another variation would be the addition of mixed numbers, such as $3\frac{1}{6} + 2\frac{1}{3}$. Note that this exercise does not require carrying, but it leads logically to those cases where carrying is needed, such as $3\frac{5}{6} + 2\frac{2}{3}$. Throughout this phase of work, efforts are made to relate this type of addition to the addition of whole numbers, since many concepts are the same for the two operations.

The next step of difficulty usually encountered is that in which the least common denominator is the product of the two denominators, as in $\frac{1}{2} + \frac{1}{5}$. This, of course, can also be shown concretely if such an approach is needed. It is probably of even greater importance,

however, to show that this kind of exercise is basically the same as those studied earlier.

Subsequently, this leads to the case in which the product of the denominators is not the *least* common denominator. When this type of exercise is introduced, the teacher has a good opportunity to show the value of using the *least* common denominator. For example, in $\frac{1}{6} + \frac{3}{8}$, we use the common denominator 24 rather than 48 primarily because the sum will not require reduction. *Would you consider it incorrect if a student used* 48 *as the denominator but handled the work correctly otherwise?*

Ultimately, the student reaches the type of exercise where no particular pattern is used, so that he is confronted with the task of getting the least common denominator with a minimum of help, as in $\frac{4}{9} + \frac{1}{6}$. Of course, he can always use the product of the denominators as a point of departure. Currently, a widely used procedure is seeing whether the larger denominator is evenly divisible by the smaller. If it is not, this same test is applied to the multiples of the larger. The smallest multiple of the larger denominator that is evenly divisible by the smaller denominator will be the least common denominator. *Can you explain why?*

In general, the approach to the treatment of unlike fractions in addition is closely paralleled by the work in subtraction. The first such case is usually the one in which the least common denominator is one of the given denominators. Frequently, this is followed by the type in which the common denominator is the product of the denominators. Later, the variation in which the product of the denominators is greater than the least common denominator is considered. In all these cases, much attention is given to work with mixed numbers. The process of borrowing receives major emphasis. Indeed, this approach has largely superseded the earlier method of using improper fractions. For example, in the exercise $3\frac{1}{4} - 1\frac{1}{2}$, earlier generations of students would probably have changed to $\frac{13}{4} - \frac{3}{2} = \frac{13}{4} - \frac{6}{4} = \frac{7}{4} = 1\frac{3}{4}$. The more common method now is $3\frac{1}{4} - 1\frac{1}{2} = 2\frac{5}{4} - 1\frac{1}{2} = 2\frac{5}{4} - 1\frac{2}{4} = 1\frac{3}{4}$. This latter approach is less mechanical and, hence, should be more meaningful than was the former method.

Multiplication. Before introducing multiplication involving fractions, it would be in order to review, at least briefly, the meaning of this process as applied to whole numbers. A student must be clear on the fact that $\begin{array}{r} 4 \\ \times 3 \\ \hline \end{array}$ means $4 + 4 + 4$, or the grouping of 3 fours into a single group, before he can apply this concept to common fractions.

Many teachers like to introduce multiplication of common fractions by using the special case of a fraction multiplied by a whole number. This case may be readily illustrated concretely. For example, using fraction cutouts, the teacher might show that $\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Then, since multiplication represents addition of like quantities, it can be shown that this is actually $2 \times \frac{1}{3}$ ("2 one-thirds"), or $\frac{2}{3}$. Another variation might be writing the problem as 2×1 third $= 2$ thirds; this brings out the close parallel between this operation and that with denominate numbers, such as 2×1 dollar $= 2$ dollars.

After a certain amount of familiarization work with multiplication of fractions by whole numbers, the students would probably be able, through skillful questioning by the teacher, to see a shorter way to carry out this type of problem. Students should observe that, in $2 \times \frac{1}{3} = \frac{2}{3}$, we can arrive at the same result by multiplying the numerator by the whole number and retaining 3 as the denominator. With additional examples to verify that this procedure works, students can evolve a pattern or rule. Note that the rule, however, is developed from understanding.

Frequently, this work is followed by the inverse operation, in which we multiply whole numbers by fractions. This operation is a little harder to visualize than its inverse. Some teachers use examples, such as that just shown ($2 \times \frac{1}{3} = \frac{2}{3}$) as a beginning point. In multiplying whole numbers, students reached the generalization that 4×2 and 2×4 yield the same product. Hence, we would expect that $2 \times \frac{1}{3}$ and $\frac{1}{3} \times 2$ would yield the same product. From this approach, the student is led to see that we are actually dividing 2 into 3 equal parts. Only after considerable work with this type of problem, however, will the student be able to see that multiplying by $\frac{1}{3}$ is equivalent to dividing by 3.

This type of example leads logically to the next phase, multiplication of fractions by fractions. In order to add reality, many types of

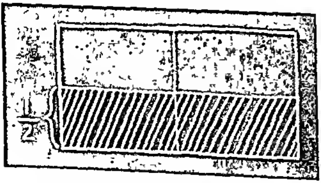

Figure 9.2

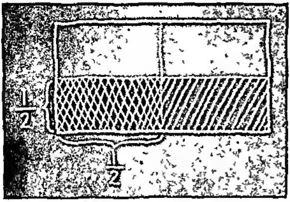

Figure 9.3

illustrative material should be used, such as folding paper, manipulating blocks, and shading drawings. For example, consider the exercise $\frac{1}{2} \times \frac{1}{2}$, or $\frac{1}{2}$ of $\frac{1}{2}$. In the block shown (Fig. 9.2), we have cross-hatched $\frac{1}{2}$ of the total area. If we designate the total area as 1, then we would describe the hatched part as $\frac{1}{2}$ of 1, or $\frac{1}{2}$. Suppose that we then take the cross-hatched part, described as $\frac{1}{2}$, and double-cross half of it; the result will be as shown. The double-marked section, then, would be $\frac{1}{2} \times$ (or *of*) $\frac{1}{2}$, or $\frac{1}{4}$ of the block. Similarly, $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, as shown (Fig. 9.3). Obviously, this type of treatment can be adapted to a number of similar examples. From these illustrations, the teacher works toward the generalization that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ by multiplying numerators and multiplying denominators of the two common fractions to get the numerator and denominator of the product *(Fig. 9.4)*.

Some texts point out that, in such exercises as $\frac{3}{5} \times \frac{1}{6}$, we can shorten the process ($\frac{3}{5} \times \frac{1}{6} = \frac{3}{30} = \frac{1}{10}$) by dividing before multiplying. This means that we could divide the 6 by the 3 to change the problem to

$$\frac{\overset{1}{\cancel{3}}}{5} \times \frac{1}{\underset{2}{\cancel{6}}} = \frac{1}{5} \times \frac{1}{2} = \frac{1}{10}$$

The 3 and 6 are said to have been *canceled.*

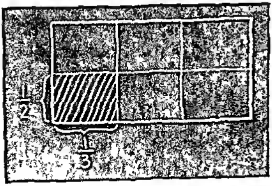

Figure 9.4

The treatment of mixed numbers varies somewhat among sixth-grade arithmetic texts. In such cases as

$$\begin{array}{r} 2\frac{1}{3} \\ \times 3 \\ \hline 1 \\ 6 \\ \hline 7 \end{array}$$

the most common procedure is to use partial products, as shown. Frequently, the same type of procedure is used for the inverse operation.

$$\begin{array}{r} 3 \\ \times 2\frac{1}{3} \\ \hline 1 \\ 6 \\ \hline 7 \end{array}$$

It is somewhat more difficult to use this method when we multiply one mixed number by another. *Can you see why?* Hence, this type of example is usually worked by converting multiplier and multiplicand to improper fractions. Thus, $2\frac{1}{2} \times 2\frac{1}{3}$ becomes $\frac{5}{2} \times \frac{7}{3} = \frac{35}{6} = 5\frac{5}{6}$. Incidentally, the teacher should be sure the students see why $2\frac{1}{2} = \frac{5}{2}$. This can easily become a mechanical and meaningless operation.

Division. If we examined the various operations with respect to the social criterion, we would probably conclude that the division of one common fraction by another functions only rarely in the life of a sixth-grade student. (How long has it been since *you* had occasion to use this process?) Yet, if we are to study arithmetic as a logical science, this operation must be included.

Historically, the division of common fractions has represented the ultimate in blind application of rules. For many years, youngsters learned to recite, "Invert the terms of the divisor and proceed as in multiplication," without any particular attention being given to the reasons why.

Several different approaches can be used in teaching division of fractions. One of these is the "incidental" or the "you can get the same answer this way" method. For example, let's consider that we have a block $\frac{3}{4}$ inch long. Using the measurement concept of division, we can readily see that we can apply a $\frac{1}{4}$-inch measure 3 times (Fig. 9.5). Hence, we would conclude, and could verify our result by actual

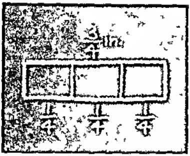

Figure 9.5

measurement, that $\frac{3}{4} \div \frac{1}{4} = 3$. After other such exercises, it is pointed out in the incidental method that we can get the same result by inverting the divisor and multiplying. Our illustration then becomes $\frac{3}{4} \div \frac{1}{4} = \frac{3}{4} \times 4 = \frac{12}{4} = 3$. It is doubtful whether this method would satisfy the curiosity of the students, especially those who frequently ask "why?"

The question is raised occasionally: "Since in multiplying fractions we multiply numerators, then multiply denominators, why don't we divide fractions by dividing numerators, then dividing denominators?" The answer is that you can, if you pick your numbers carefully. For example, in the exercise $\frac{4}{9} \div \frac{2}{3}$, it is entirely feasible to set it up as

$$\frac{4 \div 2}{9 \div 3} = \frac{2}{3}$$

You can, however, visualize how such a method would bog down if we used numbers that were not evenly divisible. Imagine trying to use $\frac{7}{8} \div \frac{2}{3}$. Then you would get

$$\frac{7 \div 2}{8 \div 3},$$

or

$$\frac{3\frac{1}{2}}{2\frac{2}{3}}$$

Another question that sometimes arises is "Why don't we use a common denominator?" Here again, the answer is that we can. Indeed, many teachers like this approach, since it makes the operation of division resemble, to some degree, the processes of addition and subtraction. Again, it is well to be sure that the numbers used for illustration will serve to clarify rather than confuse. For example, in $\frac{5}{8} \div \frac{1}{16}$, we can use a common denominator of 16, so that we have

$$\frac{10}{16} \div \frac{1}{16} = \frac{10 \div 1}{16 \div 16} = \frac{10}{1} = 10.$$

One of the most logical developments of the process used in dividing fractions, however, is based upon certain premises, each of which would need to be presented to students for their own verification: (1) If we multiply both dividend and divisor by the same number, we do not affect the quotient. For example, in $8 \div 4 = 2$, we could multiply through by 4 to get $32 \div 16$. The quotient is still 2. (2) The easiest divisor to use is 1, since any dividend divided by 1 is the unchanged dividend. Now our line of reasoning might be as follows: In $\frac{3}{4} \div \frac{3}{8}$, we could simplify the operation if we multiplied both dividend and divisor by whatever quantity is needed to yield a divisor of 1. A certain amount of experimentation will show that when any number (including a common fraction) is multiplied by its reciprocal, the product is 1. (Be sure the student knows the meaning of "reciprocal.") In our example, we would multiply both fractions by $\frac{8}{3}$, the reciprocal of $\frac{3}{8}$. Our statement now becomes $(\frac{3}{4} \times \frac{8}{3}) \div (\frac{3}{8} \times \frac{8}{3})$. Since $\frac{3}{8} \times \frac{8}{3} = \frac{24}{24}$, or 1, we could write the problem as $(\frac{3}{4} \times \frac{8}{3}) \div 1$. Now we may disregard the divisor of 1. We have inverted the divisor and changed the operation from division to multiplication. Hence, our result is 2.

Certainly, we should work toward the time when students can apply the time-honoured rule about "inverting the divisor." It is to be hoped, however, that such a rule will be evolved from an understanding of the operation. For many years, the pattern was to start with the rule and concentrate on applying it, with little or no attention given to the reasons for such a sequence of steps.

Many teachers like to begin their work in the division of fractions with the special case in which a whole number is divided by a fraction. For example, it is easy to visualize that if 2 apples are divided into halves, we get 4 halves. In symbols, $2 \div \frac{1}{2} = 4$. After working with examples such as this, students can move to a simple mixed number divided by a fraction. For example, if $1\frac{1}{2}$ apples are divided into fourths, we would get 6 such sections.

Generally, this same measurement concept can be used in introducing the division of one fraction by another. If we have $\frac{1}{2}$ of an apple, how many $\frac{1}{4}$'s of an apple do we have? This is written as $\frac{1}{2} \div \frac{1}{4}$. We can verify concretely that we get a quotient of 2.

Probably the hardest case to visualize is that in which we divide a mixed number by another mixed number. Most texts delay this process until facility in division of proper and improper fractions has been developed. Then the student is encouraged to convert the

mixed-number dividends and divisors to improper fractions. Thus, $2\frac{1}{2} \div 1\frac{1}{3}$ would become $\frac{5}{2} \div \frac{4}{3} = \frac{5}{2} \times \frac{3}{4} = \frac{15}{8} = 1\frac{7}{8}$.

Teaching aids. Among the most useful manipulative materials for this phase of arithmetic are the fraction cutouts and fraction strips which have been described earlier. All the fundamental operations with fractions can be illustrated with such materials. There are, however, many ways in which the teacher can adapt sheets of paper, rulers, fruit, and other readily available items to use in teaching fractions. In some classrooms, the apple is practically indispensable for such purposes.

Two of the Coronet films would be suited, in part, for use witb this topic. They are *Fractions: Finding the Common Denominator* and *We Discover Fractions.*

Several filmstrips could be helpful in this phase of arithmetic, among them:

"Beginning to Multiply and Divide with Fractions"
"Multiplying with Fractions"
"Dividing with Fractions"
By Filmstrip House
"Multiplying Fractions and Mixed Numbers"
"Dividing Fractions and Mixed Numbers"
By Society for Visual Education

In some of the processes with fractions, there is considerable variation among the texts in the use of terms. The teacher should examine film material rather carefully before using it in class, since differences in nomenclature could produce more confusion than enlightenment.

DECIMALS

Most of the sixth-grade programs reintroduce the study of decimal fractions by going back to the very beginning. Reference is usually made to odometers, gasoline pumps, coins, and various other devices to add reality. The equivalence between certain common fractions and decimal fractions ($\frac{1}{2} = .5$) is reviewed. Charts, place-value pockets, and similar devices are used extensively.

Addition and subtraction. After the basic structure of decimal fractions has been retaught, most sixth-grade programs go into the processes of

addition and subtraction of decimals, with special attention to tenths, hundredths, and thousandths. Some texts show an intermediate step in addition with carrying. For example, if we add $\frac{7}{10} + \frac{6}{10}$, we get $\frac{13}{10}$, or $1\frac{3}{10}$. This can also be shown by adding 7 tenths and 6 tenths to get 13 tenths, or 1 and 3 tenths. The student soon realizes, however, that it is much easier to add

$$\begin{array}{r} .7 \\ +.6 \\ \hline 1.3 \end{array}$$

Here, although he is carrying from tenths' place to ones' place, it is a straightforward and, to the student, a comparatively easy operation. Column addition involving 3 or 4 numbers with decimals up to thousandths is frequently studied in sixth grade.

Generally, students who can subtract whole numbers do quite well in subtracting decimal fractions and mixed decimal numbers, since there is a high degree of similarity between the two. Even the process of borrowing, which can be shown with place-value pockets for decimals just as it can for whole numbers, is essentially the same thing that was introduced to them several years earlier.

Multiplication. A popular method of introducing multiplication with decimal fractions is to begin with the process of addition. For example, if Henry lives 1.6 miles from school, how far does he ride his bike each day in order to make the round trip? Obviously, we can find out by adding

$$\begin{array}{r} 1.6 \\ +1.6 \\ \hline 3.2 \text{ mi.} \end{array}$$

The same result, however, can be achieved by multiplying

$$\begin{array}{r} 1.6 \\ \times 2 \\ \hline 3.2 \end{array}$$

The placement of the decimal in the introductory phases of this work is frequently done by estimating. In this example, the possible results might be listed for class discussion: .32 miles, 3.2 miles, and 32.0 miles. The "reasonable answer" tests tells us that 3.2 miles is the only one that makes sense. From a variety of such experiences, the student is led to see the generalization that we point off as many places in the product as we have pointed off in multiplier and multiplicand.

Some teachers like to relate the early phases of multiplication of decimal fractions to the comparable operation with common fractions. Thus, $1\frac{6}{10} \times 2 = 2\frac{12}{10} = 3\frac{2}{10}$. This approach might clarify the process for some students.

Students frequently have difficulty multiplying two decimal fractions. This usually is not particularly hard in decimal mixed numbers, since the sensible answer usually shows rather clearly where to place the decimal point. For example, in $2.3 \times 1.1 = 2.53$, the other possible answers (.253, 25.3, or 253) obviously would not apply. However, in such examples as

$$\begin{array}{r} .3 \\ \times .3 \\ \hline .09 \end{array}$$

how does the student decide where to place the decimal point? Obviously, one way is by applying the rule: one place in multiplier, one place in multiplicand, two places in product. Students may prefer to verify their results by using common fractions: $\frac{3}{10} \times \frac{3}{10} = \frac{9}{100}$. This can also be written as .09. Only after the student has had some experience with this type of multiplication does the product .09 look reasonable to him.

Division. The division of decimal fractions is essentially the same as the division of whole numbers. Some difficulty, however, is encountered in trying to make division with decimals meaningful. One difficulty is in deciding where to place the decimal point in the quotient.

Teachers usually begin by using a whole number as the divisor. For example, $2\overline{).8}$ rather obviously yields a quotient of .4. This can be verified by "multiplying back" as one would do with whole numbers: $2 \times .4 = .8$. The same kind of reasoning may be applied to a decimal mixed number as a dividend, such as $2\overline{)1.4} = .7$, since this is the only quotient that will yield 1.4 when multiplied by 2.

Students encounter the most difficulty when both dividend and divisor are decimal fractions. Many teachers like to use the "division as measurement" idea here. For example, in $.2\overline{).8}$ we can imagine .8 of a foot to be measured off into segments .2 of a foot in length. Obviously, our measure would be applied 4 times, or $.2\overline{).8}$ yields a quotient of 4. This result can be verified by multiplying divisor and quotient to get the dividend.

Several different methods have been used to help the student to locate the decimal point in the quotient:

1. The traditional rule is: the number of decimal places in the quotient equals the number of decimal places in the dividend less the number in the divisor. By this rule, $.5\overline{).125}$ yields .25, since three places in the dividend less one place in the divisor gives two places in the quotient. Although this system works, there is a tendency to apply it in a very mechanical manner without understanding.
2. The sensible answer system is one in which the student carries out the division process in order to find the digits in the quotient. Then he surveys the several possible decimal fractions these digits could produce. In the preceding example, these would include 25.0, 2.5, .25, .025, and others. From these, he selects one that makes sense to him and places the decimal point accordingly. This obviously requires a thorough understanding of the principles involved. A student who is deficient in such understanding would find this system to be highly unsatisfactory.
3. The multiply back system is one in which the student divides

$$\begin{array}{r} 25 \\ .5\overline{).125} \end{array}$$

without regard for the placement of the decimal point in the quotient. Then he multiplies

$$\begin{array}{r} 25 \\ \times .5 \\ \hline .125 \end{array}$$

Obviously, if .125 is the product, the multiplicand must be .25, and the decimal point is placed accordingly.
4. The equal-multiplication system. If we are to divide $.3\overline{).9}$, it would be a relatively simple matter to multiply both numbers by 10, which would yield the simple form: $3\overline{)9} = 3$. This system is rather widely used in cases where the divisor is a decimal fraction. For example, $.01\overline{)2}$ may be rewritten as $4\overline{)200}$, and the placement of the point is handled as it would be when the divisor is a whole number.

5. The caret system. This method is a rather mechanical application of the equal-multiplication system. In

$$\begin{array}{r} .25 \\ .5_{\wedge}\overline{).1_{\wedge}25} \end{array}$$

the carets are inserted so as to make the divisor a whole number, and the decimal point in the quotient is placed above the caret as indicated. This method may be applied without understanding and is not as widely used now as formerly.

Doubtless there are other variations of this general procedure for locating the decimal point in the quotient. It is impossible to point to any of them as the perfect system. Generally, the teacher will end by using the method that works best in her own classroom. It is well, however, that she realize that there are several ways of achieving the desired result.

It is important for the students to see that division with decimal fractions and mixed numbers is essentially the same process as the one they use with whole numbers. Usually, however, it is somewhat more difficult for students to visualize the operation with decimals. For example, most people find it easier to visualize a divisor of 8 than a divisor of .8. The student who understands the meanings of division will find that the processes of dividing whole numbers and decimal fractions are essentially the same.

PER CENT

Many programs in sixth-grade arithmetic do not include per cent, presumably because this topic is considered to be too advanced for students at that grade level. Others, however, introduce the per cent concept and give attention to some of the easier applications.

The study of per cent logically follows the study of decimal fractions. The earlier method, "move the decimal point two places to the right and add the per cent sign," has largely disappeared. Instead, every effort is made to give meaning to the new term.

Essentially, what is meant by *per cent*? "Per" has long been associated with the concept of rate, as in miles *per* hour, feet *per* second, and many others. "Cent" means hundred and is found in such words as "century" or "centurion" and in our own system of money. Hence, the term *per cent* means per hundred.

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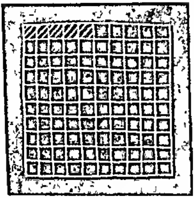

Figure 9.6

Various approaches are used to introduce the per cent concept. A hundred-chart (Fig. 9.6) is commonly used. Suppose that five squares are colored blue. The squares on the chart have been colored *at the rate of* 5 per hundred, or the rate could be described as 5 per cent. Numerous examples using the hundred-chart will help develop understanding of per cent.

Another method might be to use classroom games or school sports. Suppose that Joe won 3 of the 10 games that were played in the classroom. Students might be asked to describe this situation in a variety of ways. Some would say that Joe won $\frac{3}{10}$ of the games. If Joe's skill and luck had held out, we would expect him to win

6 games in 20
9 games in 30
15 games in 50
30 games in 100

Note that these are rates of winning. Hence, Joe won at the rate of 30 games per hundred, which can also be written as 30%. How, then, would you interpret the symbol, %?

Throughout the early phases of work with per cent, a close relationship is maintained between this system of writing fractional quantities and the two systems described earlier, that is, common fractions and decimal fractions. Hence, in the preceding illustration, the 30 per 100 idea would be written as $\frac{30}{100}$, .30, and 30% interchangeably for a time.

Those sixth-grade programs that present the per cent concept generally go little beyond an introduction. Some of them take up a few simple applications; but these are usually designed to help develop an understanding rather than to build skills in the use of per cent. If the sixth-grade student can come to see per cent as a meaningful way

of interpreting certain types of data using 100 as a reference point, he will be making good progress toward mastery of a very important and functional concept.

B. Measurement in Grade Six

Probably no area of study better illustrates the evolving nature of arithmetic than does measurement. Introduced in grade one, there is a constant reteaching and expansion of concepts each year. Many of the units formerly taught have disappeared from the texts, so that the student has fewer units and conversion factors to remember. The modern teacher, however, hopes that her students can work with measures meaningfully. This would imply that the student can (1) *estimate* with fair accuracy, (2) *visualize* situations requiring measurement, (3) evaluate the results of measurement problems, at least to the extent of deciding whether they are "reasonable" or "unreasonable."

NUMBER OPERATIONS AND DENOMINATE NUMBERS

A large majority of the units of measurement used in modern arithmetic are introduced before sixth grade. Hence, a major activity during this year is a reteaching of the units learned in earlier grades.

Considerable attention during sixth grade is given to the use of denominate numbers and interpretation of results obtained from problems using them. For example, many students fail to recognize "carrying" situations with measures. For instance, in adding the following:

4 feet	7 inches
3 feet	6 inches
7 feet	13 inches

many students would be content with the results as shown, not recognizing that they could carry a foot from the inch column. It is not unusual for students to object to our adding feet and inches in the same exercise. This is no more illogical, however, than adding dollars and cents in the same exercise. The important thing, of course, is that we add inches to inches and feet to feet. The usual system is to begin on the right, as in the foregoing exercise, and carry the foot to the left column without cluttering the situation with carry numbers. This kind of work has an incidental merit: it immediately detects any points

of confusion that might exist as to the number of quarts in a gallon, minutes in an hour, or other such relationships.

The same line of reasoning applies to subtraction. Some students find it surprising that on occasion we must borrow a gallon, a foot, or an hour. Of course, it is possible to convert all quantities to the smallest unit used, thereby avoiding any necessity for borrowing. For example,

$$\begin{array}{r} \text{8 feet 3 inches} \\ \underline{-\text{3 feet 7 inches}} \end{array}$$

could be rewritten as

$$\begin{array}{r} \text{99 inches} \\ \underline{-\text{43 inches}} \end{array}$$

This should be discouraged, however, since it is awkward and time-consuming. Rather, the student should visualize the problem as

$$\begin{array}{r} \text{7 feet 15 inches} \\ \underline{-\text{3 feet 7 inches}} \end{array}$$

This same method is the one usually recommended for all phases of subtraction with denominate numbers.

Multiplication of measures can occur under a variety of circumstances. The simplest case, however, is one in which a denominate number is multiplied by a number. For example, if it takes 2 hours 40 minutes to drive to your friend's home, how much time would be required for a round trip? The problem, of course, is

$$\begin{array}{r} \text{2 hr 40 min} \\ \underline{\times 2} \\ \text{4 hr 80 min} \end{array}$$

Note that the multiplier is 2, not 2 hours. In our application, it means that 2 trips require 4 hours 80 minutes, or 5 hours 20 minutes. Or, if a certain type of container holds 2 gallons and you have 3 such containers filled with gasoline, you have 2 gallons × 3 (in this case, number of containers), or 6 gallons of gasoline.

This is equally applicable to denominate numbers in division, except that we may have two interpretations of division: measurement and partition. If you wanted to see how many boards, each 4 feet in length, you could cut from a single board 12 feet long, you would apply the measurement concept. This means you would have 12 feet ÷ 4 feet, or one denominate number divided by another, and the quotient would not be measured in feet. Rather, it would indicate

number of boards. This may be illustrated by a cancellation type of procedure. We could write

$$\frac{12 \text{ feet}}{4 \text{ feet}}$$

and assume that units cancel in the way that numbers do, and so get a quotient of 3.

If we apply the partition idea to the same situation, our question might be: If we cut a 12-foot board into 3 equal parts, how long would each part be? Here we would have

$$\frac{12 \text{ feet}}{3 \text{ (not 3 feet)}}$$

and hence our result would be the denominate number 4 feet.

In division with denominate numbers, several different methods are possible. To illustrate: suppose that 3 gallons and 3 quarts of milk is to be divided into 3 equal parts. One way would be

$$3\overline{)3 \text{ gallons } 3 \text{ quarts}}^{\;1 \text{ gallon } 1 \text{ quart}}$$

Another method would be to write the dividend as $3\frac{3}{4}$ gallons, then divide by 3. $3\frac{3}{4} \div 3 = \frac{15}{4} \div 3 = \frac{5}{4} = 1\frac{1}{4}$ gallons. Still another way would be to write the 3 gallons 3 quarts as 15 quarts. This, when divided by 3, yields 5 quarts, or $1\frac{1}{4}$ gallons. Students who develop an understanding of denominate numbers may have considerable skill in recognizing the approach that is best for a specific problem.

AREA AND VOLUME

The use of denominate numbers in determining areas and volumes is somewhat different from that described in the previous section. In computing the area of a rectangle in square feet, we multiply two denominate numbers expressed in feet. (*Does this hold true for a circle? How?*) Conversely, if we divide 24 square feet by 12 feet, our result is 2 feet. To compute the volume of a rectangular solid in cubic feet, we multiply three denominate numbers each of which is expressed in feet. By the same kind of reasoning, we may say that a measure expressed in cubic feet divided by a measure expressed in feet yields a measure in square feet.

The commonly used measures of areas for sixth grade are the square inch, square foot, and square yard. In some programs, the square rod,

acre, section, and square mile are included, usually without major emphasis. Some sixth-grade programs include finding the volume of rectangular solids. In most texts, bowever, the study of volume is deferred until seventh grade.

Teaching aids. Several of the films and filmstrips that were cited earlier would be helpful in sixth grade. At this grade level, however, many teachers like to concentrate on the laboratory type of activity. This would require clocks, watches, and calendars for studying time; thermometers for temperature study; and scales or balances for studying weight. In working with lengths and areas, both of which are given considerable emphasis in sixth grade, the most valuable laboratory aids would be rulers, yardsticks, folding rules, and steel tapes. If available, a surveyor's chain would be valuable for working with larger areas, such as the acre.

C. Using Arithmetic in Grade Six

In the lower grades, extensive use is made of games in the teaching of arithmetic. As students advance through the grades, however, there is a decline in the amount of play activities used and a corresponding increase in the amount of attention given to real-life problems. This does not mean that fifth- or sixth-graders should be given a course in business arithmetic. In several texts, however, there has been a definite shift of emphasis in that direction.

Budgets. Although it is by no means a universal practice among sixth-grade teachers, many like to give their students some work in budget-making. Since many students at this level have an allowance or an earned income, work on budgets should be meaningful to them.

The most realistic work in budgeting would deal with the personal budgets of the students, and several texts devote some space to this topic. Another, probably less interesting, topic would be the family budget. It is doubtful whether many sixth-graders have been involved in or are interested in the family budget.

Textbooks usually limit work on budgets to basic ideas and illustrations. As is generally true, only the teacher can add reality to this work. This she can do by adapting textbook material to the community and the class and by having most class activities based upon students' budgets. There are numerous opportunities here to emphasize the

value of planning and organizing one's personal affairs. Many sixth-grade students need help and encouragement in this direction.

Other business usages. Other business applications of number are frequently introduced to sixth-graders. It should be emphasized, however, that work with these applications usually is limited to the familiarization level. Some teachers, for example, like to introduce their students to bills and receipts. Frequently, students have already had limited contact with these instruments and are interested in knowing more about them. The bill from Mary's piano teacher or John's dentist might be used to add a realistic touch. The circumstances under which one should be given a receipt and the value of receipts may be presented.

Frequently, sixth-grade texts present problems involving the concepts of profit and loss. Hence, teachers commonly give some attention to these terms. Since per cent is introduced late in sixth grade (if it is introduced at all), any work on profit or loss is usually based upon amount of profit or loss rather than on a per cent. Again, students can frequently add reality to the study of these topics by describing some of their own experiences.

Graphs and tables. Some of the most valuable work at the applied level in sixth-grade arithmetic has to do with graphs and tables. Consider how many phases of arithmetic one uses in preparing a simple bar graph. Along with work on graphs, a student frequently uses tables. Hence, these two methods of presenting data are closely associated.

Learning to read graphs is an important topic. This can be made very real to students if the graph is based upon a topic of interest to them. Frequently, students can bring to class graphs that they have found in newspapers or periodicals. If these are not too complex, they make good study materials.

It is generally true, however, that a student is not particularly successful at reading graphs until he understands their structure, and one of the best ways to develop such understanding is through practice in graph construction. Construction of graphs in sixth grade is usually limited to bar graphs and line graphs, but students are given practice in reading several other kinds, such as circle graphs, picture graphs, and the dot map.

If work with graphs and tables is to be meaningful, it is important that data of interest to sixth-graders be used. As is true of problem

work generally, the teacher must decide what kind of data would be real and interesting to sixth-graders.

SCALE DRAWINGS AND MAPS

Closely related to the construction and interpretation of graphs is the study of scale drawings. Essentially, isn't a graph a scale drawing? It is important that we remember the basic purpose of the study of graphs, maps, and scale drawings. Isn't it primarily a matter of helping students see relationships? The ratio idea should be a major part of such study.

In the student's preparation of scale drawings, it is essential that we keep in mind his level of maturity. He may lack skill in the use of some of the basic tools required, and so not be ready to produce a work of art. The assignments in the work on scale drawings should be quite simple. Frequently, the floor plan of the classroom serves as a beginning point. Later, certain parts of the playground might be used. It is doubtful, however, that sixth-graders are ready for floor plans of the school building or a house.

The chief emphasis at this grade level should be on the interpretation of drawings. This can take many forms, such as simple floor plans, a map of the school campus, a map of the town or city in which the school is located, road maps, and many others. And if a member of the group has developed skill in the construction of model airplanes or other types of models, he can be most helpful in presenting the basic ideas of scaling.

AIDS AVAILABLE

In the limited amount of work with budgets, bills, and receipts, probably the most effective materials would be the printed forms that are available. In the case of bills and receipts, the local dime store can serve as a source, since these items are normally carried in stock.

In working with scale drawings and maps, a supply of rulers is most helpful. Of course, there is never any problem in getting a supply of road maps. A map of the school campus might well be available through the principal's office; the same source might be able to supply floor plans of the building or of the classroom.

There are two Coronet films which, at least in part, would be useful at sixth-grade level. These are *Maps Are Fun* and *The Language of*

Graphs. Popular Science has several filmstrips that would be helpful in the work on money. Among these are "Earning Your Money," "Paying Your Bills," and "Spending Your Money."

D. Problem Solving in Grade Six

In general, the teaching of problem solving in the sixth grade follows the same patterns that were used in earlier grades. After each type of operation has been presented, there are drill exercises and problems applying the principles that have been learned. Numerous variations are used, such as problems without numbers, problems with insufficient data, problems with an excess of data, and incomplete problems in which the student provides the questions. Also used is the straightforward type of problem where a situation is presented and a question asked.

Most of the texts concentrate on problems which should be of interest to sixth-graders. Some illustrative problem topics are grocery shopping, planning a trip, buying gasoline, painting a room, vacations, and earning and spending money. It should be repeated, however, that the teacher should try to adapt problem situations to fit the local community. It is hoped that the textbook problems will at least be supplemented by other problems based upon local school and community. Why not let the students do some problem making along with problem solving?

USING LETTERS OR SYMBOLS FOR NUMBERS

Although most of the sixth-grade texts refrain from using the term *algebra*, many of them do introduce some of the algebraic concepts. One way is in the use of letters or symbols ("unknowns") for numbers in certain types of work. Actually, this will not be new to the students since, as early as third grade, they have encountered such exercises as $3 + \square = 7$.

Some of the "missing number" types of problems or exercises for sixth grade are

$$6 = 5\frac{?}{12} \quad \text{and} \quad 4 = 3\frac{?}{6}$$

$$\frac{8}{10} = \frac{?}{100} \quad \text{and} \quad \frac{7}{8} = \frac{?}{24}$$

$$8\frac{3}{4} = 8\frac{?}{8} = 7\frac{?}{8} \quad \text{and} \quad 6\frac{2}{3} = 6\frac{?}{9} = 5\frac{?}{9}$$

14 quarts = ? gallons ? quarts

25 days = ? weeks ? days

4 yards 1 foot = 3 yards ? feet

$$16 = \frac{2}{3} \text{ of } ? \quad \text{and} \quad 8 = \frac{3}{4} \text{ of } ?$$

$$\frac{9}{10} \text{ of } ? = 81$$

Many teachers use the missing number approach in finding common denominators or in the reduction of fractions. For example, to write $\frac{3}{4}$ as twelfths, we might set it up as

$$\frac{3}{4} = \frac{?}{12}$$

It is to be hoped that such an operation would be meaningful, not just an application of a mechanical process.

FORMING GENERALIZATIONS

The ability to think mathematically is a skill that is somewhat difficult to develop. Some teachers find that they can further the process by encouraging students to form generalizations.

This, of course, amounts to encouraging students to evolve patterns, rules, or formulas that would help solve a particular type of problem. The traditional approach, still widely used, has been for the teacher, or teacher and student cooperatively, to follow the textbook presentation of a process. This usually led to a rule or formula which was given at the end of the explanation. The task of the student was to follow (hopefully, with understanding) the presentation, then apply the rule.

In another approach, the student is given certain kinds of data and attempts to develop a pattern or rule. For example, many sixth-grade classes deal with problems involving distance, rate, and time. Some of the students in an average sixth-grade class would, under proper guidance, be able to set up the basic formula, distance = rate × time, based upon their own experience and observation. This approach is more challenging than simply presenting the formula, followed by pages of problems using it.

It is doubtful, however, that all the students in a sixth-grade class would profit by the approach just described. In some cases, we would be happy if the student involved would learn to work even the most routine problems. Unfortunately, this is frequently true of students who are above average in other subject areas. And it sometimes happens that a student who does quite well in arithmetic exercises cannot seem to handle problems.

E. Some Departures from Earlier Patterns

During the late 1950's, several of the national organizations of mathematicians and mathematics teachers became concerned over the mathematics curriculum in the elementary and secondary schools. Particularly, they expressed fear that students were seeing mathematics as "the frozen product of antiquity" rather than as a "living and ever-growing subject." Several universities served as study centers for groups who were looking into the problem. Later, writing groups were created, their function being to produce curricular materials to be used on an experimental basis at various grade levels.

One such group has worked on study materials in mathematics for grades four, five, and six. Some of the topics proposed for inclusion in the mathematics curriculum at the intermediate level are

1. A concept of sets and operations with sets
2. Numeration, including considerable attention to number systems with bases other than 10
3. Principles and techniques of addition and subtraction; the abacus is featured in this work
4. Sets of points, this serving to introduce some aspects of geometry
5. Principles and techniques of multiplication and division
6. Concept of fractional number
7. Linear measure
8. Factors, primes, and common denominators
9. Triangles
10. Measurement of angles
11. Area
12. The integers
13. Exponents.

Even a casual examination of the topics listed will indicate that the sponsoring group is suggesting a rather drastic departure from the usual content. Many topics are being introduced earlier than was formerly the case. Further, all the material is presented from the point of view of the mathematician. Doubtless this approach to elementary mathematics will be tested and evaluated over a period of years. The work of these groups is likely to have a definite influence on the context of elementary mathematics programs.

F. The Exceptional Learner

Although, as teachers, we must accept the fact that the "average student" does not exist, it is true that a large number of our students progress according to the same general pattern. It is a matter of common sense that we devote considerable time and effort to teaching this group.

It is also important, however, that we be aware of those students who depart from the "normal" level of achievement. Every effort should be made to meet the needs of the rapid learners and the slow learners. This cannot be done if all our teaching is directed entirely to the "average."

THE SLOW LEARNER

It requires very little effort on the part of the teacher to locate the students who are having trouble. But it does require much careful attention if the causes of difficulties are to be recognized. Diagnosis of learning difficulties is an important field.

Some of the principles that apply in working with the slow learner have already been discussed. They may be summarized as follows:

1. Through use of diagnostic tests, close observation of written work, examination of procedures through "work aloud," and by any other device available, try to locate the student's specific difficulties.

2. Work toward the correction of these difficulties by using explanations and practice which will apply specifically to the difficulties. This, of course, must be individualized. Probably, this is one of the biggest problems from the viewpoint of the teacher.

Many teachers observe that, even at sixth-grade level, the greatest single source of error is an inadequate knowledge of the basic number

combinations. Teachers should look for opportunities to give the slower students more drill on the basic facts using the four fundamental operations, with special emphasis on those facts that are known to give trouble.

THE RAPID LEARNER

The student who can master the prescribed arithmetic program and have time left over deserves an opportunity to deal with challenging material, not in lieu of, but in addition to, the work done by other students.

Many of these students would cherish an opportunity to proceed on their own in using enrichment materials. A good collection of such material is available. For example, the previously cited series by Harper & Row has the following titles for sixth grade: "Arithmetic of Long Ago," "Short Cuts to Multiplying," "Excursions in Arithmetic," "Crossnumber Puzzles," "Brain Teasers," "Faster and Faster," "Amusing Problems," and "Some Curious Numbers."

Although designed primarily for older students, the Webster Publishing Company's series, "Exploring Mathematics on Your Own," probably would appeal to some sixth-graders. Some of the titles are "Sets, Sentences and Operations," "Fun With Mathematics," "Understanding Numeration Systems," and "Short Cuts in Computing."

Some sixth-grade teachers like to keep a few high school mathematics texts on hand for students who want them. Such materials would probably be of occasional interest to some of the more rapid learners in the sixth-grade group. Usually, these would be for browsing and for familiarization work, not for making assignments.

Something to Think About

1. Observe a sixth-grade student as he works on arithmetic. Prepare a list of items you noted which might be significant in diagnosis of difficulties.
2. Some of your sixth-grade students add downward in column addition. Others add upward. Would you have them change to your system? Why?
3. Your class is working on missing terms in the various operations. One student objects because her parents told her "algebra is for ninth grade." How would you deal with this situation?

4. Make a list of the problems that might result if you constantly asked a rapid learner to tutor a slow learner. Can you find references dealing with this situation?
5. In your opinion, what is the most important single objective in sixth-grade arithmetic? Defend your choice.
6. A student suggests that you are inconsistent in that you permit him to multiply feet by feet but object when he multiplies quarts by quarts. How would you explain your position?
7. What would you consider to be appropriate remedial work for a student who (1) counts fingers, (2) mumbles, (3) consistently misses multiplication problems involving carrying?

Selected References

Adler, Irving. *The Magic House of Numbers.* New York: The John Day Company, Inc., 1957.
Good enrichment material for the middle grades.

Archer, Allene. *Number Principles and Patterns.* Boston: Ginn and Company, 1961.
Interesting enrichment material for intermediate grade levels.

Marks, John L., James R. Smart, and Irene Sauble. *Extending Mathematical Ideas.* New York: Ginn and Company, 1961.
An enrichment work-text that would be appropriate for sixth grade.

School Mathematics Study Group. *Mathematics for the Elementary School.* New Haven. Yale University Press, 1961.
Contains suggestions regarding content in arithmetic for grades four, five, and six.

Swain, Robert L. *Understanding Arithmetic.* New York: Holt, Rinehart & Winston, Inc. 1957.
Also very good on common fractions.

Thiele, C. L. "Arithmetic in the Middle Grades," *Fiftieth Yearbook*, Part II, The National Society for the Study of Education. Chicago: University of Chicago Press, 1951.
Gives some excellent material on working with fractions.

Number Operations for the Teacher

CHAPTER TEN / NUMBER OPERATIONS FOR THE TEACHER

Definitions of several "kinds" of numbers and operations on these numbers have been made in previous chapters. The kinds of numbers include the natural numbers, integers, rationals, and real numbers. Traditionally, arithmetic as an elementary school subject has been concerned with operations on these kinds of numbers by making use of the Hindu-Arabic, base-10, positional system of numeration. Also, some characteristics of these numbers, some processes peculiar to our system of numeration, and some applications of these numbers have been presented.

More recently, emphasis has been placed on the structure of our number systems (we have more than one) and on the nature of our systems of numeration. Understanding is facilitated by making clear distinctions between numbers and numerals, between number operations and renaming processes with numerals, and between characteristics of systems of numbers and characteristics of systems of numerals.

This chapter presents some concepts of number operations and numerals, organized as follows:

A. *Number Structure*

B. Operations on Fractions

C. Operations on Decimals

D. *Other Number Concepts*

A. Number Structure

The base-10, positional system of numeration has already been explained. Also, we indicated that this is only one of several systems by which numbers may be written or named. Our understanding of number is closely related to our understanding of the system of numeration we use. So close is this relationship that we frequently come to confuse numbers with their names. We commonly refer to the symbol for a number as the number itself. If the difference is understood, no harm is done. The nature of numbers, numerals, and number operations should be clear to the teacher and, eventually, to the student.

NUMBER OPERATIONS

What is a number operation? In our more or less formal development of the several kinds of numbers, operations on numbers were defined. Our first numbers, the cardinal or natural numbers, were developed from elementary concepts of sets. Also, the operations of addition and multiplication of natural numbers were defined in terms of sets. Our first numbers and number operations, then, grew out of basic conceptions of sets of objects and the regrouping of sets of objects to form new sets. These definitions had nothing to do with a system of numeration or a way of symbolizing numbers.

Upon accepting a system of numeration for the natural numbers and some symbols to indicate number operations, we are confronted with such symbols as $56 + 25$, 13×36, $34 - 18$, and $87 \div 16$. We commonly refer to these symbols as indicating a number operation. Now, "$56 + 25$" is the name of a number. What number? We are expected to find a standard name for the number. How do we do this? The number $56 + 25$ in our standard form is 81. We get this by "adding." In other words, *addition* in arithmetic has come to mean the process by which we rename a number to some standard form. This was not our original definition of *addition*. The operation of addition is a number operation. The process by which we rename an indicated number to a standard form is peculiar to our system of numeration.

For example, $56 + 25$ needs to be renamed to 81. How do we get the 81? First, we must know the so-called basic addition facts. These we get by counting. Then we must know that, in our positional system of numeration,

$$56 = 50 \text{ and } 6 \quad \text{or} \quad 5 \text{ tens and } 6 \text{ ones}$$

and

$$25 = 20 \text{ and } 5 \quad \text{or} \quad 2 \text{ tens and } 5 \text{ ones}$$

Now, we use our basic number facts to add digits in corresponding positions to get 7 tens and 11 ones, which are regrouped to get 8 tens and 1 one, which in our system of numerals is written 81. It should be clear now that to go from the number $56 + 25$ to its standard form 81 is a process peculiar to our system of numerals. In Roman numerals, for example, the process would be very different. This process of renaming is what we have come to think of as addition. Similar renaming procedures involving multiplication, subtraction, and division signs may be detailed.

PROCESSING NUMERALS

The term *processing numerals* has been applied to the procedure for finding the standard name of an indicated number. This is done to emphasize the point that the "meaning" of the operations on numbers lies in the definition of the operation and the "laws" or rules obeyed by the operation. These definitions and laws, of course, have nothing to do with any particular system of numeration or symbols.

The more traditional approach to arithmetic presents number operations and the renaming processes with the Hindu-Arabic numerals as synonymous concepts. Such an approach leaves the impression that addition is defined by the procedures whereby one finds the standard name for an expression such as $56 + 25$.

So far as computational skill of the child is concerned, these rather fine distinctions are not likely to matter. If, on the other hand, we have the more remote aim of developing future understanding and application of number to more difficult and abstract mathematical concepts, then these distinctions should perhaps gradually evolve in a planned arithmetic program.

NUMBER STRUCTURE

The teacher at any grade level should know exactly what number system is under consideration. In the primary grades, the natural numbers are used. These are:

$$0, 1, 2, 3, 4, 5, \ldots, 75, 76, 77, \ldots$$

In the middle elementary grades, the non-negative rational numbers are studied. These include such numbers as:

$$0, \quad 1, \quad \tfrac{1}{2}, \quad \tfrac{3}{2}, \quad 6, \quad 1\tfrac{2}{3}, \quad \tfrac{8}{10}$$

Also introduced are the nonnegative real numbers or decimals. Examples are

$$0, \quad 1.0, \quad 4.2, \quad .666\ldots, \quad 7.125, \quad 3.14\ldots$$

Both finite and infinite decimals are included. In the upper elementary grades, the positive and negative integers and operations on them are considered. Concepts of negative integers may then be extended to include negative rational and real numbers.

Prime numbers. The theory of numbers is the study of the cardinal or natural numbers. One of the oldest divisions of such numbers was into even and odd numbers. Characteristics of odd and even numbers were discussed in an earlier chapter along with their relation to the set of all natural numbers.

A very important concept is that of *prime* numbers. A prime number is a number greater than 1 not divisible by any other number than itself and 1. The first few prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, \ldots$$

There are twenty-five primes less than 100. Numbers that are not prime are called *composite.* Note that 2 is the only even number that is prime.

Factors. In the set of natural numbers, if $a \times b = c$, then a and b are called factors of c. For example, $3 \times 6 = 18$, thus 3 and 6 are factors of 18. Also, 2 and 9 are factors of 18, as well as 1 and 18. Thus, the factors of 18 are 1, 2, 3, 6, 9, 18. If we allowed rational numbers to be factors, 18 would have an infinite set of factors. This points up the importance of knowing what system of numbers is under consideration at any time.

We frequently need to know the prime factors of a number. The prime factors of 18 are 2 and 3. Note that $2 \times 3 \times 3 = 18$. If we wish to find the prime factors of a number, we need try only those primes whose squares are equal to, or less than, the number. For example, to find whether 89 is a prime, we need to try dividing 89 by 2, 3, 5, and 7. Since 11 squared is greater than 89, we know that if 89 had a prime factor greater than 11, then it would have one less than 11. *Why?*

Fundamental theorem of arithmetic. Any natural number greater than one may be factored into primes in only one way. For example, $36 = 4 \times 9$, but 4 and 9 are not prime. If we factor 4 and 9, we get

$$36 = 2 \times 2 \times 3 \times 3$$

Also, $36 = 3 \times 12$. The number 3 is a prime, but 12 has factors 2 and 6. Thus, $36 = 3 \times 2 \times 6$, which may be written $36 = 3 \times 2 \times 2 \times 3$. These are the same prime factors as found for the first instance except in order. This rather obvious theorem or statement will not be proved here.

Greatest common divisor. Frequently, in considering two numbers, we need to find a number which will divide each of the numbers. This number is called a *common divisor* or *factor*. Often we want the greatest common divisor. Now, we know that any number may be expressed as a product of primes in only one way. Also, the product of any prime factors of a number is a divisor of the number. We may use this idea to find the greatest common divisor of two numbers. For example, what is the greatest common divisor of 180 and 315? Factor each number into prime factors.

$$180 = 2^2 \times 3^2 \times 5$$
$$315 = 3^2 \times 5 \times 7$$

The factors common to 180 and 315 are 3^2 and 5. The greatest common divisor is 45.

Now, if numbers are large they may be difficult to factor. The euclidean algorithm provides a way of finding the greatest common divisor without factoring. Suppose we wish to find the greatest common divisor of 165 and 286. First divide the larger number by the smaller number.

$$\begin{array}{r} 1 \\ 165\overline{)286} \\ \underline{165} \\ 121 \end{array}$$

Then divide the divisor by the remainder, repeating each time until a remainder of 0 is reached.

$$\begin{array}{r} 1 \\ 121\overline{)165} \\ \underline{121} \\ 44 \end{array} \qquad \begin{array}{r} 2 \\ 44\overline{)121} \\ \underline{88} \\ 33 \end{array} \qquad \begin{array}{r} 1 \\ 33\overline{)44} \\ \underline{33} \\ 11 \end{array} \qquad \begin{array}{r} 3 \\ 11\overline{)33} \\ \underline{33} \\ 0 \end{array}$$

The divisor last used is the greatest common divisor of the two numbers. In this case, it is 11.

Find the greatest common divisor of 154 and 195.

$$\begin{array}{r} 1 \\ 154\overline{)195} \\ 154 \\ \hline 41 \end{array} \qquad \begin{array}{r} 3 \\ 41\overline{)154} \\ 123 \\ \hline 31 \end{array} \qquad \begin{array}{r} 1 \\ 31\overline{)41} \\ 31 \\ \hline 10 \end{array} \qquad \begin{array}{r} 3 \\ 10\overline{)31} \\ 30 \\ \hline 1 \end{array} \qquad \begin{array}{r} 10 \\ 1\overline{)10} \\ 10 \\ \hline 0 \end{array}$$

The greatest common divisor of 154 and 195 is 1. Later we will make use of the greatest common divisor to reduce fractions to "lowest terms." This is what we have defined as putting rationals in standard form.

Least common multiple. The least common multiple of a pair of natural numbers is the smallest natural number divisible by both numbers. The prime factors help us determine the least common multiple. For example, find the least common multiple of 12 and 30.

$$12 = 2^2 \times 3$$

$$30 = 2 \times 3 \times 5$$

The least common multiple contains all the different prime factors of both numbers and contains them as many times as they most appear in either number. Thus,

$$2^2 \times 3 \times 5 = 60$$

is the least common multiple of 12 and 30. The addition and subtraction procedures for fractions make use of the least common multiple.

B. Operations on Fractions

Our logical development of numbers led to a system of numbers called *rationals*. Operations were defined for this set of numbers. The *rationals* were defined as an ordered pair of integers. Therefore, each ordered pair of integers is a rational number. How are these rational numbers related to our common fractions?

FRACTIONS AND RATIONALS

In our logical development of the rationals, we found there was justification for considering the symbol $\frac{a}{b}$ to mean division of a by b.

The more common meaning given to fractions in the elementary grades, that of an implied division of natural numbers, is only one of several interpretations that may be given rationals. Fractions may be thought of as ratios or *relative comparisons* of two sets. The symbol $\frac{a}{b}$, read "a over b" or "a divided by b", is not restricted to the integers. We may let a and b themselves be fractions. The line drawn between two fractions indicates division. Thus we have

$$\frac{2}{3}, \quad \frac{\frac{1}{2}}{\frac{3}{4}}, \quad \frac{1+6}{3+7}, \quad \frac{\frac{3}{4}-\frac{1}{3}}{\frac{2}{3}+\frac{5}{6}}$$

all written in fraction form.

We should think of a fraction as a number among an ordered set of numbers. The numeral for a fraction $\frac{a}{b}$ is composed of two parts separated by a line. The part above the line is called the *numerator*, and the part below the line is called the *denominator*.

Earlier, we defined a standard form for a fraction. We call this *reducing a fraction to its lowest terms*. This simply means that if $\frac{a}{b}$ is a fraction, and if a and b have no common divisor other than 1, then $\frac{a}{b}$ is in standard form. Thus we have a family of fractions represented by the member in standard form. Note that if we are given any member of the family, say $\frac{c}{d}$, we may multiply or divide both numerator and denominator by the *same* number and get another member of the family. For example,

$$\frac{3}{4} = \frac{3 \times 12}{4 \times 12} = \frac{36}{48}$$

and

$$\frac{35}{77} = \frac{35 \div 7}{77 \div 7} = \frac{5}{11}$$

Members of the same family of fractions are "equal." (Recall our definition of equal rationals.) This means that, in any operation involving a fraction, we may substitute for the fraction any member of the family represented by the fraction.

Fractions less than 1 are called *proper* fractions. Examples are $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{13}$, $\frac{127}{231}$. Fractions greater than 1 are called *improper* fractions. Examples

are $\frac{11}{7}$, $\frac{4}{3}$, $\frac{13}{2}$. Every improper fraction may be considered as the sum of an integer and a proper fraction, in which case it is called a *mixed* number. For example,

$$\frac{11}{7} = 1 + \frac{4}{7} \quad \text{or} \quad 1\tfrac{4}{7}$$

Since we may think of a fraction as an indicated division, divide 7 into 11 to get the number 1 with a remainder of 4. We indicate the division of 4 by 7 as $\frac{4}{7}$. Thus the quotient of 11 divided by 7 is $1\frac{4}{7}$.

ADDITION AND SUBTRACTION OF FRACTIONS

The addition and subtraction of fractions are, of course, defined by our definitions of the addition and subtraction of rationals. Most texts separate addition and subtraction of fractions into two cases involving "like" and "unlike" fractions. This really refers, not to the fractions, but to the numerals denoting the fractions. Thus $\frac{5}{7}$ and $\frac{3}{7}$ are like fractional numerals, since their denominators are the same numeral, whereas $\frac{3}{7}$ and $\frac{5}{14}$ are unlike fractional numerals, since the denominators are not the same numeral. Note, however, that "$\frac{3}{7}$" and "$\frac{6}{14}$" represent the same fraction.

We are again faced with a question. Is addition an operation on abstractions called *fractions* or is addition a renaming process on number names or numerals? For all practical purposes, in the elementary grades the latter is what we mean by addition of fractions.

If the fractional numerals of two fractions are "like"; that is, they have the same denominator, then we add the numerators and write this number over the denominator to get the fractional numeral which represents the sum of the fractions. Thus,

$$\frac{5}{16} + \frac{4}{16} = \frac{9}{16}$$

We subtract the numerators and write this number over the denominator to get the difference between the fractions. Thus,

$$\frac{5}{16} - \frac{4}{16} = \frac{1}{16}$$

If the fractional numerals are unlike, we change them to like fractional numerals. We find the least common multiple of the two denominators, then denote the fractions as fractional numerals with the least common

multiple as the denominator. For example, we may add $\frac{1}{6}$ and $\frac{7}{8}$. The least common multiple for 6 and 8 is 24. Then,

$$\frac{1}{6} = \frac{1 \times 4}{6 \times 4} = \frac{4}{24}$$

$$\frac{7}{8} = \frac{7 \times 3}{8 \times 3} = \frac{21}{24}$$

We now have like fractional numerals, and add accordingly. To subtract $\frac{3}{10}$ from $\frac{5}{6}$, we must find the least common multiple of 10 and 6, which is 30. Then

$$\frac{5}{6} = \frac{5 \times 5}{6 \times 5} = \frac{25}{30}$$

$$\frac{3}{10} = \frac{3 \times 3}{10 \times 3} = \frac{9}{30}$$

and

$$\frac{25}{30} - \frac{9}{30} = \frac{25 - 9}{30} = \frac{16}{30}$$

The standard form for $\frac{16}{30}$ is $\frac{8}{15}$. *Why?*

Mixed numbers may be changed to improper fractions by simply adding the integer to the proper fraction. Thus, $4\frac{5}{6}$ means $4 + \frac{5}{6}$. The rational 4 means $\frac{4}{1}$, so we have

$$\frac{4}{1} + \frac{5}{6} = \frac{4 \times 6}{1 \times 6} + \frac{5}{6} = \frac{24}{6} + \frac{5}{6} = \frac{29}{6}$$

We may add mixed numbers by changing them to improper fractions and adding, as has been indicated, or we may simply add the integral parts and fractional parts of the mixed numbers separately. For example, we may add $2\frac{1}{3}$ and $3\frac{7}{8}$. Changing to improper fractions, we get

$$2\tfrac{1}{3} = \frac{6}{3} + \frac{1}{3} = \frac{7}{3}$$

and

$$3\tfrac{7}{8} = \frac{24}{8} + \frac{7}{8} = \frac{31}{8}$$

Upon adding, we get

$$\frac{7}{3} + \frac{31}{8} = \frac{7 \times 8}{3 \times 8} + \frac{31 \times 3}{8 \times 3} = \frac{56}{24} + \frac{93}{24} = \frac{149}{24} = 6\tfrac{5}{24}$$

We may add as follows:

$$
\begin{array}{rl}
2\tfrac{1}{3} & = 2 + \dfrac{8}{24} \\
+3\tfrac{7}{8} & = 3 + \dfrac{21}{24} \\
\hline
 & 5 + \dfrac{29}{24} = 5 + 1\dfrac{5}{24} = 6\dfrac{5}{24}
\end{array}
$$

MULTIPLICATION AND DIVISION OF FRACTIONS

Multiplication of fractions is defined by our definition of the multiplication of rationals. If $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

The product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators. We need to consider multiplication when we have fractional numerals of different kinds. We may have a whole number 4 which, as a fraction, means $\frac{4}{1}$. We may have a proper fraction, such as $\frac{3}{5}$, and we may have improper fractions or mixed numbers, such as $\frac{8}{3}$ or $2\frac{2}{3}$. What is $4 \times \frac{3}{5}$? Write this as a fraction multiplied by a fraction:

$$\frac{4}{1} \times \frac{3}{5} = \frac{4 \times 3}{1 \times 5} = \frac{12}{5}$$

What is $\frac{3}{5} \times \frac{8}{3}$? This is $\frac{24}{15}$. Then we have the case $4 \times 2\frac{2}{3}$. We may write this as

$$4 \times 2\tfrac{2}{3} = 4 \times \frac{8}{3} = \frac{32}{3}$$

In this case, we change the mixed number to an improper fraction and multiply.

Division was defined as the inverse of multiplication. If $\frac{a}{b}$ and $\frac{c}{d}$ are rationals, then

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

This is the same result as if we "inverted" the divisor $\frac{c}{d}$ and multiplied, that is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Note that $\frac{d}{c}$ is the reciprocal of $\frac{c}{d}$. Thus, to divide by a fraction, we multiply by its reciprocal. For example,

$$\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$

SOLVING PROBLEMS WITH FRACTIONS

Most problems applying fractions to practical situations may be recognized as one of three types. An understanding of these types will help the teacher and student to decide how to solve a problem.

The first type of question asked in a problem involving fractions is as follows: What number is $\frac{3}{4}$ of 36? This may be written

$$\underline{\quad ? \quad} = \frac{3}{4} \times 36$$

There is a second type of question: The number 27 is what fractional part of 36? This may be written

$$27 = \underline{\quad ? \quad} \times 36$$

The third type of question is: The number 27 is $\frac{3}{4}$ of what number? We write this

$$27 = \frac{3}{4} \times \underline{\quad ? \quad}$$

Each of these types of problems involves a simple equation in which we wish to find the missing number. Names have been given to the three numbers involved in the equation.

$$27 = \frac{3}{4} \times 36$$

fractional part = fraction × base

Upon reading a problem involving fractions, the student must decide which two of the foregoing numbers are given. The objective is to find the missing number. Solutions are indicated as follows:

Fractional part missing:

fractional part = fraction × base

Fraction missing:

$$\text{fraction} = \text{fractional part} \div \text{base}$$

Base missing:

$$\text{base} = \text{fractional part} \div \text{fraction}$$

Note that the fractional part may be larger than the base. Also, both the fractional part and the base may themselves be fractions. The three types of problems with fractions may be understood by relating them to the multiplication of whole numbers. If the multiplier, multiplicand, and product of whole numbers are understood, then the relation of fraction, base, and fractional part should be clear.

C. Operations on Decimals

We have seen how the real numbers were defined in terms of repeating and non-repeating decimals. Further, it was shown that all rationals or common fractions may be written as repeating decimals or as terminating decimals if we do not wish to repeat zero. The term *decimal fraction* is used to mean a special group of common fractions, those whose denominators are one of the following:

$$10, 10^2, 10^3, 10^4, \ldots$$

We sometimes shorten the term *decimal fraction* to simply decimal.

Not every one writes decimals as we do. We commonly use a *decimal point* in our decimal numerals, placed even with the bottom of the symbols, as in 26.365. In England, the point is written higher up, as in 2·63; in other European countries, a comma is used as the separator in decimals.

PLACE-VALUE STRUCTURE

Decimal fractions are written simply as an extension of the Hindu–Arabic base-10 system of notation. Each succeeding place to the left of the ones place increases by a factor of 10, and each succeeding place to the right of the ones place decreases by a factor of $\frac{1}{10}$. Thus, the place value in a decimal numeral is as follows:

	hundreds		*ones*		*hundredths*	
1000	100	10	1.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
thousands		*tens*		*tenths*		*thousandths*

The number 8356.472 means

$$8(1000) + 3(100) + 5(10) + 6 + 4\left(\frac{1}{10}\right) + 7\left(\frac{1}{100}\right) + 2\left(\frac{1}{1000}\right)$$

Note that the "center" of the decimal numeral is the ones place, not the decimal point. The point simply tells us which is the ones place.

The place-value system for writing decimal numerals makes the processing of decimal numerals very much like the processing of whole numbers or integers. We may analyze the numeral representing a decimal number several ways. For example, some ways that 56.27 may be analyzed are as follows:

5 tens + 6 ones + 2 tenths + 7 hundredths

56 ones + 2 tenths + 7 hundredths

56 ones + 27 hundredths

FRACTIONS TO DECIMALS

Earlier, we presented a method by which rationals could be expressed as a terminating decimal or as an infinite repeating decimal. Any common fraction written in lowest terms whose denominator has prime factors including only 2's and 5's may be written as a terminating decimal, that is, a decimal fraction. Thus, we have

$$\frac{1}{4} = \frac{25}{100} = .25$$

$$\frac{4}{5} = \frac{8}{10} = .8$$

$$\frac{3}{16} = \frac{1875}{10{,}000} = .1875$$

Other common fractions when converted into decimals do not terminate, but instead give a repeating decimal. These may be written as decimal approximations, using as many places as are desired. Thus, $\frac{1}{3}$ may be written as .33 if two places are accurate enough, or as .333 if accuracy to the thousandth place is desired.

DECIMALS TO FRACTIONS

Any terminating decimal may be expressed as a common fraction by removing the decimal point and writing the numerator and denominator. Examples are:

$$.5 = \tfrac{5}{10}, \quad .75 = \tfrac{75}{100}, \quad 3.167 = 3\tfrac{167}{1000}$$

If the numerators are divisible by 2 or 5, these fractions may be reduced to lower terms.

The method for converting infinite repeating decimals to common fractions has been demonstrated. Infinite non-repeating decimals have no common fraction equivalents.

Some frequently used common fraction–decimal equivalents are

$\frac{1}{2} = .5$	$\frac{1}{3} = .33\frac{1}{3}$
$\frac{1}{4} = .25$	$\frac{2}{3} = .66\frac{2}{3}$
$\frac{3}{4} = .75$	$\frac{1}{6} = .16\frac{2}{3}$
$\frac{1}{8} = .125$	$\frac{1}{12} = .08\frac{1}{3}$
$\frac{3}{8} = .375$	$\frac{1}{5} = .2$
$\frac{5}{8} = .625$	$\frac{3}{5} = .6$
$\frac{7}{8} = .875$	$\frac{4}{5} = .8$
$\frac{1}{10} = .1$	$\frac{1}{100} = .01$

Some of the preceding fractions give infinite repeating decimals. These have been written as a combination of a decimal and fractional numeral. For $\frac{1}{3}$, we have $.33\frac{1}{3}$ in which the $\frac{1}{3}$ means $\frac{1}{3}$ of $\frac{1}{100}$.

OPERATIONS WITH DECIMALS

In adding or subtracting whole numbers, we write the numbers with the places aligned, that is, ones under ones, tens under tens, and so on. No new principle is involved in adding or subtracting decimals. Simply align places when the numerals are written and add or subtract as whole numbers. For example, to add 26.75 and 7.6, we write

$$\begin{array}{r} 26.75 \\ +7.6 \\ \hline 34.35 \end{array}$$

The usual rule for multiplying decimal fractions is to multiply as if the decimals were whole numbers and "point off" as many decimal places in the product as there are decimal places in the multiplier and

multiplicand together. The rule may be demonstrated. Find the product of 1.6 and .32. Following the rule, we get

$$\begin{array}{r} .32 \\ \times 1.6 \\ \hline 192 \\ 32 \\ \hline .512 \end{array}$$

If we change these decimals to fractions, we have

$$1.6 = 1\tfrac{6}{10} \qquad \text{and} \qquad .32 = \tfrac{32}{100}$$

Then

$$1\tfrac{6}{10} \times \tfrac{32}{100} = \tfrac{16}{10} \times \tfrac{32}{100} = \tfrac{512}{1000}$$

Note that we multiply 16 and 32 to get the numerator, and 10 and 100 to get the denominator. To change the fraction $\frac{512}{1000}$ to a decimal, we point off just as many places in the 512 as there are tens in the denominator to get .512.

Multiplication of a decimal fraction by 10 may be demonstrated as follows: The number 43.76 means

4 tens + 3 ones + 7 tenths + 6 hundredths

Suppose we move the decimal point one place to the right to get 437.6. This is

4 hundreds + 3 tens + 7 ones + 6 tenths

Each place has been increased by a factor of ten, and the number is ten times as large as before. If the decimal is moved to the left, we have 4.376. Then we have

4 ones + 3 tenths + 7 hundredths + 6 thousandths

Each place is one-tenth as large as before and so the number is one-tenth as large. Therefore, to multiply a decimal by 10, simply move the decimal point one place to the right in the decimal numeral. To divide by 10, move the decimal point one place to the left in the decimal numeral.

Division of decimals. Every division example involving decimals may be written as a decimal divided by a whole number. Suppose we wish to divide 37.5132 by 8.6. In our operations with fractions, we could multiply numerator and denominator by the same number and get an equivalent fraction or a member of the same family of fractions. We expressed this by saying that the "value" remained unchanged. We

may think of a division example as a fraction. Thus, we may multiply divisor and dividend by 10 or any power of 10 so that the divisor becomes a whole number. The quotient remains unchanged by this operation. For example, $37.5132 \div 8.6$ is the same number as $375.132 \div 86$. We divide as we would whole numbers or integers, ignoring the decimal point.

```
      4.362
 86)375.132
    344
     311
     258
      533
      516
       172
       172
```

The rule is as follows: If the places in the dividend and quotient are carefully aligned, place the decimal point in the quotient in the place above the decimal point in the dividend. *Why this rule?*

The rule should be meaningful. Why is the above answer 4.362? We could determine an approximate answer that would help us decide where to put the decimal point. The divisor is roughly 100 and the dividend is roughly 400; therefore, the quotient is approximately 4. We can be fairly sure the quotient is not 43.62 or .4362. The decimal point should be placed to give a quotient near the approximate quotient 4.

We could change the decimals to common fractions, divide, and change back to decimals. Thus, we would have $37\frac{5132}{10,000} \div 8\frac{6}{10}$. Other than to demonstrate a few examples this is not feasible. Decimals were invented to shorten the tedious task of multiplying and dividing large common fractions.

Various kinds of examples in dividing decimals may be encountered. If the moving of decimal points in dividend and divisor is understood and if the rule for placing the decimal point in the quotient is understood, no trouble should be experienced. Find the quotient of $10.35 \div .045$. We write this

```
.045)10.35
```

We need to multiply the dividend and divisor by 1000 to make the divisor a whole number. (Moving the decimal point 3 places to the

right does this.) It is necessary to annex a zero to the dividend to get 10,350 ÷ 45. Dividing, we get

$$
\begin{array}{r}
23 \\
45\overline{)10350.} \\
\underline{90} \\
135 \\
\underline{135}
\end{array}
$$

Now, is the answer 23? No, we place the decimal point in the quotient according to the rule and fill the vacant place with a zero to get 230 as the quotient. Why? Again we find the approximate answer. Let the divisor be 50 and the dividend be 10,000; then the quotient is 200 because $50 \times 200 = 10{,}000$. Our answer, then, must be 230, instead of 23. To show that a whole number is a decimal, we may write this answer 230.0.

Decimal fractions are terminating decimals. How do we perform operations with infinite decimals? If the infinite decimals are repeating decimals, we may find the equivalent common fractions, perform the operation, and change back to decimals, writing as many places as desired. We may simply terminate the infinite decimal at the desired degree of accuracy and then perform operations as though they were decimal fractions.

SOLVING PROBLEMS WITH DECIMALS

Problems using decimals are of three types, as are those involving common fractions. These will be illustrated by examples. We have the following relation in such problems:

$$\text{fractional part} = \text{decimal fraction} \times \text{base}$$

The first type occurs when the fractional part is missing. What number is .65 of 250? We write this:

$$\underline{\ \ ?\ \ } = .65 \times 250$$

We find the fractional part to be 162.5 by multiplying.

The second type of problem has the decimal fraction missing. What decimal fraction of 45 is 9? We write

$$9 = \underline{\ \ ?\ \ } \times 45$$

or

$$\underline{\ \ ?\ \ } = 9 \div 45$$

The decimal fraction is .2.

The third type of problem has the base missing. The number 12 is .3 of what number? We have

$$12 = .3 \times \underline{\quad ? \quad}$$

or

$$\underline{\quad ? \quad} = 12 \div .3$$

The base is 40. Knowing these three cases, we need to decide in a given problem which items are given and which item is to be determined. Other problems involving increase or decrease in numbers may have other steps combined with one of these three types.

D. Other Number Concepts

Several other number concepts are useful in number applications. Raising numbers to powers and extracting roots have almost become basic operations. Whereas our four basic number operations are binary, these two operations are uniary. They are operations on a single number. Ratio and proportion and per cent are rather standardized concepts.

POWER AND ROOTS

A rather brief presentation of powers and roots has been made. We have seen how our base-10 positional number system is based on powers of 10. When a number is used as a factor several times, we may indicate this by a special symbol. Thus,

$$10 \times 10 \times 10 = 10^3$$

$$2 \times 2 \times 2 \times 2 = 2^4$$

The number used as a factor several times is called the *base*. The superscript to the right of the base is called the *exponent* and indicates how many times the base is used as a factor. A kind of arithmetic of exponents may be set up.

Now $3^2 \times 3^4$ means $(3 \times 3) \times (3 \times 3 \times 3 \times 3)$

The associative law of multiplication permits us to group factors any way we choose; therefore,

$$(3 \times 3) \times (3 \times 3 \times 3 \times 3) = 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6$$

Note that the exponent of the product is the sum of the exponents of

the original powers of 3. We may state this as a rule. Let a be the base and m and n exponents. Then,

$$a^m \times a^n = a^{m+n}$$

Suppose we have $2^6 \div 2^4$. Then,

$$2^6 \div 2^4 = \frac{2 \times 2 \times 2 \times 2 \times 2 \times 2}{2 \times 2 \times 2 \times 2} = 2^2$$

The rule for division of powers is

$$a^m \div a^n = a^{m-n}$$

where $m > n$.

Now, suppose we have a case in which $n > m$, such as $4^3 \div 4^5$. This means

$$4^3 \div 4^5 = \frac{4 \times 4 \times 4}{4 \times 4 \times 4 \times 4 \times 4} = \frac{1}{4^2}$$

Now, if we let $4^3 \div 4^5$ mean 4^{3-5}, then we get 4^{-2}, a negative exponent. We will then define 4^{-2} to mean $\frac{1}{4^2}$. We may let the division rule hold for all integers m and n, with the understanding that

$$a^{-m} = \frac{1}{a^m}$$

Suppose $m = n$, and we have $a^m \div a^n$, then

$$a^m \div a^n = a^{m-n} = a^0$$

but if $m = n$, then $a^m = a^n$ and $a^m \div a^n = 1$. We then define any base raised to the zero power to be 1. The definitions of zero and negative integers as exponents make them obey our rules for exponents.

The base may be a fraction. What is $(\frac{3}{4})^3$? This means

$$\left(\frac{3}{4}\right)^3 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{3 \times 3 \times 3}{4 \times 4 \times 4} = \frac{3^3}{4^3}$$

To raise a fraction to a power, we raise the numerator and denominator to the power.

The inverse operation to raising to a power is extracting a root. The square root and cube root algorithms will not be demonstrated. However, some conventions regarding notation are desirable.

The symbol "$\sqrt{a}$" means the square root of a, the symbol "$\sqrt[3]{a}$"

means the cube root of a, and so on. The root of a number may be indicated by a fractional exponent. Thus,

$$\sqrt{a} = a^{1/2}, \quad \sqrt[3]{a} = a^{1/3}, \quad \sqrt[4]{a} = a^{1/4}$$

If we have the root of a number raised to a power such as $\sqrt[4]{a^3}$, we write

$$\sqrt[4]{a^3} = a^{3/4}$$

where the numerator of the exponent tells the power to which a is raised, and the denominator tells the root extracted.

We may have powers of powers, such as $(2^2)^3$. This is read "2 squared to the third power." What does this mean?

$$(2^2)^3 = 2^2 \times 2^2 \times 2^2 = (2 \times 2) \times (2 \times 2) \times (2 \times 2)$$
$$2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$$

The rule $(a^m)^n = a^{m \times n}$. The rule for extracting the nth root of a power is $(a^m)^{1/n} = a^{m/n}$ or a^{m-n}. For example, find the square root of 2^6. We write this

$$\sqrt{2^6} = (2^6)^{1/2} = 2^{6/2} = 2^3$$

Scientific notation. Extremely large or small numbers may be written in a more compact way using powers of 10. We may note that the sun is 93,000,000 miles from the earth, or the wavelength of a microwave generator is .0000015 centimeter. These are, of course, approximations, each with two significant digits. (See section on measurement in Chapter 13.) We may write these numbers in scientific notation by writing them as a number equal to, or greater than, 1 but less than 10, multiplied by an integral power of 10.

We simply shift the decimal point so that one digit stands to the left of the point and multiply by the power of 10 necessary to move the point back to its original position. To write 93,000,000 in scientific notation, move the decimal point to the left seven places to get 9.3. Multiply this number by 10^7 to indicate the original position of the decimal point.

$$93{,}000{,}000 = 9.3 \times 10^7$$

We may make use of our negative exponents in writing small numbers in scientific notation. To change .0000015 to scientific notation, we move the decimal point to get 1.5, then multiply by 10^{-6} to indicate its original position. This is the same as dividing by 10^6. So

$$.0000015 = 1.5 \times 10^{-6}$$

This notation makes very large or small numbers with only a few significant digits easier to understand. Computation is easier. Suppose you desired to find the product of 93,000,000 and .0000015. Our regular multiplication process would be rather long. Consider the following.

$$
\begin{aligned}
(9.3 \times 10^7) \times (1.5 \times 10^{-6}) &= (9.3 \times 1.5) \times (10^7 \times 10^{-6}) \\
&= (9.3 \times 1.5) \times 10^{7-6} \\
&= (9.3 \times 1.5) \times 10^1 \\
&= 9.3 \times 1.5 \times 10 \\
&= 13.95 \times 10 = 139.5
\end{aligned}
$$

Most of the foregoing steps could be done mentally. They have been included to illustrate the process. The placement of the decimal point is facilitated by use of the scientific notation.

RATIO AND PROPORTION

Traditional texts usually define ratio as an implied division. The ratio of the number 12 to the number 4 is $12 \div 4$ or $\frac{12}{4}$. Also, we write 12:4, and this is read "the ratio of 12 to 4." This definition is very nearly the same as the traditional definition of common fractions. Some texts note that fractions "express" a ratio and treat fractions and ratios as being equivalent. One might wonder why they should be treated separately if they are really the same thing.

Many kinds of problems in everyday affairs involve comparisons or rates. We drive 240 miles and burn 15 gallons of gasoline, so we are getting 16 miles to the gallon. We averaged 45 miles per hour for 6 hours, so we are 270 miles from home. Our electricity costs us $1.85 per thousand kilowatt hours. The meter says we have used 9250 kilowatt hours. *How much is the bill?* Ratio has been defined as a division, a rate, a comparison, or a fraction. Is there a general concept of ratio that would encompass all of these ideas?

Suppose we use ordered pairs of numbers again. This time, let the numbers be real numbers. Remember the real numbers have subsets isomorphic to the rationals, the integers, and the natural numbers. Examples of such number pairs are (1, 4), ($\frac{1}{2}$, 2), (2, 2), (3.6, 1.8).

We define a *ratio* to be an ordered pair of numbers. The ordered pair (a, b) as a ratio will be written $\frac{a}{b}$ or $a:b$, and will be read, "a is to b." The ratio $\frac{a}{b}$ and the ratio $\frac{c}{d}$ are members of the same family of ratios if $ad = bc$. Two members of the same family of ratios are said to be *equal* and a statement that they are equal is called a *proportion*. Thus,

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion. Consider the ratio

$$\frac{\sqrt{2}}{2} \text{ and } \frac{1}{\sqrt{2}}$$

Are these ratios equal? (Members of the same family.) Does

$$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}?$$

How does our definition of ratio differ from our definition of rationals?

Numerous applications of proportion may be made. In all such cases, we have given a ratio and one number which is part of a ratio equal to the given ratio. The problem is to find the other number in the equal ratio. Our definition of equal ratios supplies the answer. If $\frac{a}{b}$ and $\frac{c}{d}$ are equal ratios, then $ad = bc$. Suppose the numbers a, b, and d are given. We wish to find c. We may let x represent the unknown number. Set up the proportion

$$\frac{a}{b} = \frac{x}{d}$$

Then $ad = bx$. We then solve this equation for x. Now suppose c is known and d is missing. We have the proportion

$$\frac{a}{b} = \frac{c}{x}$$

and $ax = bc$. Solve this equation for x. These are the two cases possible in a proportion example.

In such problems, the given ratio may be expressed as a fraction, a rate, or a comparison. In many cases, it is convenient to use an equation

called a *formula*. For example, $d = rt$ means distance equals rate multiplied by time. The rate is a ratio. Distance to time is a ratio. These are equal; therefore, we have a proportion. Our rate may be 45 miles per hour. This is the ratio $\frac{45}{1}$. Our time is 6 hours. We have the proportion

$$\frac{45}{1} = \frac{\text{distance}}{6\text{ hours}}$$

Distance $\times 1 = 45 \times 6$. Our distance is 270 miles. The formula was simply a proportion written in a convenient form. It is an equation already solved for your convenience.

In a general sense, we may describe a family of ratios as one kind of *relation* sometimes called a *proportionality relation*. In general, we may define a *relation* as a set of ordered pairs of numbers. This is a convenient device to describe more advanced number concepts, such as functions.

PER CENT

Where common fractions have denominators that are powers of 10, we may omit the denominator and write the common fraction as a decimal fraction. The decimal fraction indicating hundredths has come to have numerous applications. Through common usage it has become convenient for us to partition things into 100 parts and then consider the parts. A number indicating hundredths has come to be called a *per cent*.

The bank charges us 8 per cent interest per year on a loan of \$300.00. This means we are charged interest at the rate of 8 parts to each 100 parts of the loan. Thus, a per cent is a ratio in which the second of the ordered pair of numbers is understood to be 100. We write a per cent as a number followed by the word "per cent" or the symbol "%." Thus, 8 % means $\frac{8}{100}$ or .08.

We have written ratio in the same form as common fractions. We should remember however, that a common fraction is a number, whereas a ratio is a relation. Even so, most texts take up the matter of converting common fractions to per cents and decimal fractions to per cents. These texts usually have defined fractions, not as numbers, but as an implied division of two numbers or as a ratio.

Applications of common fractions and decimal fractions make it convenient to think of them as ratios. To convert common or decimal

fractions to per cents, write them as fractions with 100 for the denominator. The numerator becomes the per cent. Examples:

$$\frac{1}{2} = \frac{50}{100} = 50\%$$

$$\frac{3}{4} = \frac{75}{100} = 75\%$$

$$\frac{5}{4} = \frac{125}{100} = 125\%$$

$$.62 = \frac{62}{100} = 62\%$$

$$.769 = \frac{76.9}{100} = 76.9\%$$

$$5.86 = \frac{586}{100} = 586\%$$

$$\frac{2}{3} = \frac{66\frac{2}{3}}{100} = 66\tfrac{2}{3}\%$$

In some cases, the numerator is not a whole number when the denominator is 100. In these cases, the numerator may be a fraction or mixed number, or it may be a decimal fraction or mixed decimal. For example,

$$\frac{1}{3} = \frac{33\frac{1}{3}}{100} \text{ or } \frac{33.3+}{100}$$

$$= 33\tfrac{1}{3}\% \text{ or } 33.3+\%$$

The plus sign indicates an approximation.

We are also concerned with changing per cents to fractional or decimal form. The per cent sign indicates hundredths. When we remove it, we must divide by 100. For common fractions, we simply write $25\% = \frac{25}{100}$. To convert to decimal fractions, we move the decimal point two places to the left. *Why?* Thus, $25\% = .25$. Examples:

$$10\% = \frac{10}{100} = \frac{1}{10} \quad \text{and} \quad 10\% = .10$$

$$50\% = \frac{50}{100} = \frac{1}{2} \quad \text{and} \quad 50\% = .50$$

$$250\% = \frac{250}{100} = \frac{5}{2} \quad \text{and} \quad 250\% = 2.50$$

$$12.5\% = \frac{12.5}{100} = \frac{125}{1000} = \frac{1}{8} \quad \text{and} \quad 12.5\% = .125$$

$$66\tfrac{2}{3}\% = \frac{66\frac{2}{3}}{100} = \frac{200}{300} = \frac{2}{3} \quad \text{and} \quad 66\tfrac{2}{3}\% = .66\tfrac{2}{3}$$

Problems in per cent. Since per cents are ratios, problems involving per cents are proportions with one of the four numbers missing. Example: What number is 25% of 60? The given ratio is 25% or $\frac{25}{100}$. We need to find the equal ratio $\frac{x}{60}$, where the x is to be determined. We have

$$\frac{25}{100} = \frac{x}{60}$$

$$100x = 25 \times 60 = 1500$$

$$x = \frac{1500}{100} = 15$$

The answer is 15. It has been convenient to develop a formula for proportions involving per cents. It is

$$\text{percentage} = \text{rate} \times \text{base}$$

$$p = rb$$

In the preceding example, the rate is 25%, the base is 60, and the percentage is 15. The similarity between per cent problems and those involving common and decimal fractions is evident. When any two (of the percentage, rate, and base) are given, the third may be computed. In the computation, the rate is always changed from a per cent to a common or to a decimal fraction. In our example, we changed 25% to $\frac{25}{100}$. We could use the formula and change 25% to .25. This would give us

$$\begin{aligned} \text{percentage} &= r \times b \\ &= .25 \times 60 \\ &= 15 \end{aligned}$$

The formula may be "solved" for any one of the three components:

$$p = rb$$

$$r = \frac{p}{b}$$

$$b = \frac{p}{r}$$

Many per cent problems involve an increase or decrease in a number. We need to decide which number is the base. For example: The cost of an article was $12.00. The article was marked to sell for $15.00. What per cent profit was made? The cost of the article is the base, whereas the profit, $3.00, is the percentage. The rate, then, is found by substituting in

$$\begin{aligned} rate &= \frac{p}{b} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

The rate or per cent profit is $\frac{1}{4}$, or 25%.

A third type of per cent problem is that of finding the base when the rate and percentage are known. This type of problem does not have as many applications as the other types, and some texts advocate not teaching it in the elementary grades. An example: A paper boy saves 40% of his earnings. At the end of the first year he has saved $120.00. What were his earnings for the year? The rate is 40%; the percentage $120.00. Therefore, we have

$$\begin{aligned} base &= \frac{p}{r} \\ &= \frac{120}{.4} \\ &= 300 \end{aligned}$$

His earnings for the year were $300.00.

Per cents are constantly used in business in problems involving interest, discounts, commissions, profit and loss, net price, selling price,

margin, and cost. Such fields as economics, finance, and the social sciences make frequent use of per cent. Its numerous applications to everyday affairs make per cent an important concept in arithmetic.

Something to Think About

1. Is a number operation something we do with numbers or is it something we do with numerals? Explain.
2. Can you explain what is meant by the term *number structure*?
3. How do prime numbers differ from other numbers?
4. Select some numbers at random. Find whether they have a greatest common divisor other than one. Use the factoring method and the euclidean algorithm.
5. Show how most of our *rules* for operations on fractions are really *rules* for processing fractional numerals.
6. Explain how our *rule* for the division of fractions really comes from our definition of the multiplication of rationals.
7. Explain the place-value method of writing decimal fractions.
8. Explain the meaning of negative exponents.
9. What is meant by the scientific notation of numbers? What are its advantages? Make up examples of numbers that could be expressed in scientific notation.
10. Explain several ways of defining ratios.
11. Why is proportion such an important concept in arithmetic?
12. Using the percentage formula, give examples of the three types of per cent problems. How may we generalize proportion and per cent into a pattern that will aid in solving problems?

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PART FOUR / MATHEMATICS IN THE UPPER GRADES

Mathematics in Grade Seven

The junior high school grades have been referred to as the "lost years" so far as mathematics is concerned. Traditional programs have reviewed computational procedures and applications to various business problems. Since seventh-grade students are ten years or more away from securing loans, mortgages, and insurance, these applications are largely forgotten by the time they are needed. Recently, many changes have been suggested in the arithmetic programs of the seventh and eighth grades. The general trend of these proposals is to introduce more basic mathematical topics and eliminate many of the repetitious applications of arithmetic.

This chapter presents the following topics:

A. Testing and Diagnosis
B. A Seventh-grade Program
C. Problem Solving
D. The Exceptional Learner

A. Testing and Diagnosis

In traditional arithmetic programs, most arithmetic concepts had been presented by grade six. Grades seven and eight were used to reteach and extend understandings and applications of concepts already introduced. More recently, many groups are recommending the introduction of new mathematical content in grades seven and eight. The evaluation procedures carried on by the teacher in these grades will then depend in part upon the arithmetic content of these grades. *Evaluation* is used here in the broad sense and includes a consideration of objectives, content, and various appraisal procedures, including testing and diagnosis.

No attempt will be made to formulate objectives for grades seven and eight. Many excellent statements of mathematical objectives may be found in the literature.[1] The introduction of new content in these grades may tend to stress the mathematical outcome rather than the social outcomes. Accordingly, our appraisal and the procedures related to appraisal will change.

Four types of tests are available to the upper grade arithmetic teacher. These are the standardized tests of achievement in arithmetic, ready-made diagnostic tests of achievement, ready-made tests found in arithmetic texts, and teacher-made tests. Standardized tests have a place in a program of evaluation. National norms provide a basis for comparison, and many of these tests are rather comprehensive. Some of the most frequently used standardized tests suitable for grades seven and eight are

1. Analytical Scales of Attainment in Arithmetic (California Test Bureau)
2. California Achievement Test (California Test Bureau)
3. Coordinated Scales of Attainment (Educational Test Bureau)
4. Iowa Every Pupil Tests of Basic Skills, Advanced Examination, Basic Arithmetic Skills (State University of Iowa)
5. Metropolitan Achievement Test (Harcourt, Brace and World)
6. New Stanford Achievement Examination (Harcourt, Brace and World, Inc.,)
7. SRA General Achievement Tests (Science Research Associates)

[1] Leo Brueckner, *et al.*, *Developing Mathematical Understanding in the Upper Grades* (New York: Holt, Rinehart & Winston, Inc., 1957), chap. 2.

Diagnosis for difficulties in arithmetic should continue in grades seven and eight. For the traditional topics in these grades, ready-made diagnostic tests may be used. (See Chapter 5 for a list of such tests.)

Tests in texts or teachers' manuals may be used for diagnostic purposes. Teacher-made tests probably must be used to diagnose learning difficulties or measure achievement in the new content that is coming into these grades.

A broader view of evaluation becomes necessary as our objectives for arithmetic instruction change. Increased emphasis on mathematical understandings may require that we adopt new methods of appraisal. Harding[2] notes that functional situations may be used in broadened programs of evaluation. A list of twenty-three content topics is provided to illustrate the need for teachers to be fully acquainted with arithmetic content. Applications of this content may be used to evaluate the concepts and skills of the student for each topic.

The teacher should use many techniques in evaluating the work of students. Some or all of these techniques should be employed during the school year:

1. Standardized tests of achievement
2. Ready-made diagnostic tests in one or more areas of arithmetic
3. Inventory tests in texts, workbooks, or teachers' editions of texts
4. Teacher-made tests
5. Observation
6. Records of student participation in class activities
7. Individual conferences
8. Functional applications of arithmetic concepts

B. A Seventh-grade Program

The problems of selecting topics and deciding how far they should be developed in a particular grade are not likely to be solved permanently. The changing needs and demands of society make it necessary constantly to review and evaluate arithmetic and other programs in our schools. The program

[2] Lowry H. Harding, "Emerging Practices in Evaluation of Elementary School Mathematics," *Emerging Practices in Mathematics Education*, Twenty-Second Yearbook (Washington, D.C.: National Council of Teachers of Mathematics, 1954), pp. 355–63.

proposed here may need additions and deletions to fit the needs of a school or community. Nevertheless, the emerging mathematics program for the junior high school grades does have recognizable features. Some attempt has been made to include these features in this program.

RETEACHING AND EXTENSION

Most of the basic facts concerning the nature of numbers and operations on whole numbers, common fractions, and decimals are introduced before grade seven. These concepts need to be reviewed or retaught where necessary. Some kind of inventory or achievement test may be used to evaluate a new class and make a decision about reteaching. Then some extension of these basic concepts should be made.

Nature of Numbers. By the time a student reaches the seventh grade, he should have gained rather definite concepts of our decimal number system and the Hindu-Arabic system of numerals. The student should understand the difference between a number and a numeral. As for the whole numbers, the student should demonstrate some understanding that a number is an abstract concept denoting plurality or manyness or quantity. He should know that a numeral is a symbol for a number just as a noun is a symbol for a person, place, or thing. The collection or set concept and the one-to-one correspondence concept may be used to associate a number with a collection of objects.

The most important feature of our system of numeration is place value. Since ours is a decimal system, the value of a digit one place to the left is increased ten times. Zero as a place holder, indicating no digits in a place, made possible a concise written system of numeration. Before the use of zero, some mechanical place holder, such as the abacus, had to be used to indicate places. The absence of a bead indicated the absence of a digit.

The teacher should be sure students understand the meaning of the decimal feature of our numerals. This may be done by having them tell the value of digits in a large numeral. For example, in the numeral 568,312, they may be asked to tell what value the 8 has or the 1 or the 5. Students should clearly understand that 568,312 means:

$$5(100,000) + 6(10,000) + 8(1,000) + 3(100) + 1(10) + 2$$

Some arithmetic texts emphasize that a digit has two values: one is its *face* value which is based on its meaning as a number, the other is its *place* value which is based on its position in a written numeral. Thus in

the numeral "63" the "6" has a face value of six and a place value of ten. In the numeral "444" each "4" has a face value of four, but place values of hundred, ten, and one from left to right.

Operations with whole numbers. The seventh-grade teacher must be prepared to review and extend understanding of number operations with whole numbers. It is entirely possible for students to exhibit skill in computation with whole numbers, but lack the understanding necessary for successfully proceeding to more advanced topics in mathematics.

Students at this level are not yet ready for a completely abstract treatment of the number processes. The basic ideas of sets of objects can be very useful in developing understanding of the so-called fundamental operations. The number operations may be considered a process of grouping and re-grouping. Numbers may be re-grouped in many ways. Thus 42 may be thought of as 4 tens and 2 ones, as $50 - 8$, or as $30 + 12$, or in many other ways. Students should clearly understand that adding means putting two or more groups together to make one group. Subtraction should be understood as the inverse of adding and means taking apart or separating a group into two subgroups, one of which is known. Teachers should indicate that "borrowing" in subtraction is really re-grouping. Thus to subtract 18 from 42, we regroup 42 into $30 + 12$ rather than $40 + 2$.

Multiplication should be understood at this level as an operation which combines several equal groups into one group. Re-grouping may also be used to simplify multiplying. Thus to multiply 23 by 12, students think of 12 as 10 and 2. Ten 23's would be 230 and two 23's would be 46, and 230 and 46 are 276.

Dividing should be understood as separating a group into a given number of equal subgroups or into groups of a given size. This may be demonstrated as the inverse of multiplying. The two concepts of division are sometimes called *measurement* and *partition*. For example: How many 3's are there in 18? We could subtract 3 repeatedly to find how many 3's are in 18. This is the measurement idea. If 18 is divided into 6 equal groups, how large is each group? This is an example of the partition idea. Students should discover the relationship among divisor, quotient, and dividend in division examples.

Laws of operation. The laws or rules regarding whole numbers and the operations of addition and multiplication may be introduced, though perhaps not by name. Thus, students should discover the

commutative and associative laws of addition and multiplication. The order in which numbers are added makes no difference, and the order in which two or more numbers are multiplied does not affect the product.

Also, they should discover that the indicated sum of two or more numbers is multiplied by a number when each number in the indicated sum is multiplied by the number. Thus:

$$6(3 + 4) = 18 + 24$$

and likewise

$$5(7 - 2) = 35 - 10$$

Now, an indicated product of two or more numbers is multiplied by a number when only one number or term in the indicated product is multiplied by the number. This fact is sometimes confused with the distributive rule just cited.

Common fractions. Many students entering the junior high school grades have only a hazy idea of common fractions. Much of their knowledge centers around using fractional numerals in the four fundamental processes. An abstract presentation of rational numbers cannot be made at this level. Some kind of concrete representation of common fractions should be made. Common fractions may be presented as parts of a whole, parts of a group, as a ratio or comparison, and as an indicated division. Fractions as ratios or comparisons are not so easily understood by children as the first two concepts.

A concept of *equal* fractions should be developed. Thus we say that $\frac{3}{4} = \frac{12}{16}$. These are said to be *equal* or *equivalent* fractions. A fraction chart can be used to demonstrate equivalent fractions. We may say that $\frac{1}{2} = \frac{4}{8}$.

This is demonstrated by the fraction chart (Fig. 11.1). Other concrete demonstrations may be used.

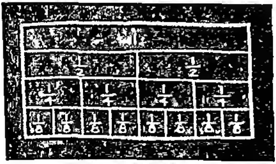

Figure 11.1

Many texts define *like* fractions. The teacher should realize that this term applies to the way a fraction is written. In other words, two fractions may be written so that the denominators of the numerals of the two fractions are the same. Thus $\frac{1}{2}$ and $\frac{2}{3}$ may be written as $\frac{3}{6}$ and $\frac{4}{6}$. The "likeness" refers to the written symbol only, and makes easier the operations on fractions.

The four operations applied to common fractions may need to be reviewed in the seventh grade. Teachers should observe the ability of students to "think through" the operations on fractions. Where understanding is lacking, illustrations and examples to develop meaningful concepts of operations on common fractions should be provided.

Decimal fractions. Decimals as *real numbers* is an abstract concept that should be understood by the teacher, but not necessarily by the students. Only certain decimals are presented as decimal fractions. These are the finite decimals; those which may be made equivalent to common fractions with denominators of 10, 100, 1000, 10,000, and so on. Decimals that are equivalent or equal to common fractions may be demonstrated.

Name		*Common Fraction*		*Decimal*
one tenth	=	$\frac{1}{10}$	=	.1
one one-hundredth	=	$\frac{1}{100}$	=	.01
one one-thousandth	=	$\frac{1}{1000}$	=	.001

The principle of place value in our numeral system may be demonstrated by a chart.

hundreds	tens	ones	tenths	hundredths
100	10	1.	$\frac{1}{10}$ or .1	$\frac{1}{100}$ or .01

The number 346.72 means

$$3(100) + 4(10) + 6 + 7(\tfrac{1}{10}) + 2(\tfrac{1}{100})$$

Powers of 10 should be understood at this grade level. Thus we may write the preceding number

$$3(10^2) + 4(10) + 6 + 7(\tfrac{1}{10}) + 2(\tfrac{1}{10})^2$$

The units place should be the center of a decimal numeral. The decimal point is used to indicate the ones place. In some countries, the point is placed above the units or ones place. The operations on decimals should be reviewed at this level as needed. Previous chapters have illustrated ways of presenting these operations.

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Graphs. The various kinds of graphs are introduced before grade seven. In the seventh grade, students should understand the difference between a line graph and a bar graph. Students should observe that a line graph is a continuous line connecting points determined on a coordinate system. The bar graph is a discontinuous graph made up of bars of various lengths.

Students should be introduced to a coordinate system. The number line should be reviewed and pupils shown how points and numbers correspond. A coordinate system may be illustrated as a pair of perpendicular number lines. Illustrate how the point (3, 2) is located (Fig. 11.2). Explain that units along the number lines are entirely arbitrary and may be chosen to fit the data to be graphed. Explain how points on the plane correspond to number pairs, with the first number telling the units out on the horizontal axis and the second number telling the units up on the vertical axis.

Social applications. Traditional texts have placed much emphasis on social applications. The topics usually included in the seventh grade have been budgets, buying, banks, interest, accounts, business, discounts, and profit and loss. Many of these so-called social applications are very far removed from the everyday life of the seventh-grade student. Even though some of the advantages attributed to these applications, such as immediate application and interest, may have been far-fetched, some good can come from broad social applications of number concepts. More applications in concrete situations can help students generalize number concepts. A problem-solving approach may be emphasized in these applications.

The teacher should continue to make use of social applications as

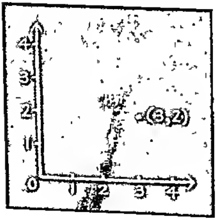


Figure 11.2

new number concepts are introduced or old ones reviewed. Thus problems involving whole numbers, common and decimal fractions, and the four operations on these numbers should be taken up as learning is reviewed and extended. Problems involving ratio and proportion and per cent have numerous applications. Various topics involving measurement furnish many everyday problem situations. In each case, teachers should present new terms or ideas peculiar to the application and then emphasize a problem-solving approach. The most important outcomes may be, not the applications themselves, but an understanding that mathematics is a series of structures and that problems fall into patterns. When a pattern is known many problems can be easily solved. Learning mathematics then is learning patterns or structures and finally reaching a point when new patterns may be evolved.

RATIO AND PROPORTION

One of the most frequently encountered ideas in elementary mathematics is that of comparing numbers. If we wish to find how much larger or smaller one number is than another, subtraction is involved. This is comparison by subtraction. If we wish to find *how many times* larger or smaller one number is than another, then division is involved. This is comparison by division.

A ratio is a comparison of two numbers by division. Example: A mother is 40 years old, and her daughter is 10 years old. Compare the mother's age to the daughter's age by division.

$$40 \div 10 \quad \text{or} \quad 40/10 = 4$$

The mother's age is 4 times the age of her daughter. The ratio is 4 to 1 or 4:1 or 4/1. Now the comparison may be phrased as follows: Compare the daughter's age to the mother's age by division:

$$10 \div 40 \quad \text{or} \quad 10/40 = 1/4$$

The ratio is now 1 to 4. This may be written 1:4 or 1/4.

Where measurements are to be compared they must be stated in the same units. Inches may be compared to inches, but not to feet, yards, or miles. Examples using common measures should be used to illustrate this point.

Problems involving ratio form an important part of the applications of comparisons to everyday life. Formerly such problems were taught

as proportion, and a mechanical rule was applied to get the answer. Now emphasis is placed on understanding ratio and thinking through a problem. Example: If 4 apples cost 21¢, how much are apples per dozen? The student should see that a comparison is involved. It is 4 to 12 or 4/12. The fractional form for writing the ratio 4/12 is 1/3, reduced to simplest terms. Thus 1/3 of a dozen costs 21¢. A dozen costs 3×21¢ or 63¢. (Fig. 11.3.)

A proportion shows the equality of two ratios. The preceding problem may be expressed as follows:

$$4/12 = 21/? \quad \text{or} \quad 1/3 = 21/?$$

If an x is put in place of the question mark, we have an equation.

$$1/3 = 21/x$$

Students should know how to change fractions to equivalent or equal fractions. So the student should reason that 21 times the numerator of 1/3 gives the numerator of the equal fraction. Then he must multiply the denominator of 1/3 by 21 to get the denominator of the equal fraction.

$$\frac{1 \times 21}{3 \times 21} = \frac{21}{63}$$

The ratio of 4 apples to a dozen apples is 1 to 3, the ratio of the cost of 4 apples to the cost of a dozen apples is 21 to 63. The cost of a dozen apples is 63¢. Solving proportions as equations may come after some elementary concepts of algebra have been introduced.

Some texts now advocate introducing ratio as a *number-pair*, then defining a *proportionality relation* between number-pairs. This approach is more in line with newer concepts of numbers that are being introduced in the elementary grades. This is obviously a relatively abstract presentation of ratio and proportion. Experimentation with these concepts may be valuable to the teacher.

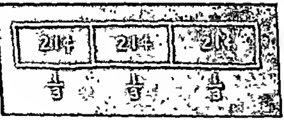


Figure 11.3

PER CENT

The meanings of per cent and the changing of common and decimal fractions to per cents may be introduced before grade seven, but a redevelopment of the basic meanings of per cent is usually needed. A chart or board with 100 squares is very useful in demonstrating the 100 basis for per cent (Fig. 11.4). Examples may be used to show the equivalence of fractions with denominators of 100 and per cents. Out of many such examples can come an understanding of the equivalency of common fractions and per cents. Thus,

$$25/100 \text{ is } 25 \text{ per cent or } 25\%$$

But 25/100 may also be written as a decimal

$$25/100 = .25$$

Therefore, .25 is 25 per cent or 25%. The fact that

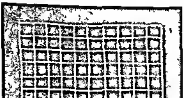
$$25/100 = .25 = 25\%$$

can be developed from the hundred board. This relationship needs to be generalized so that students may easily convert from common fractions or decimal fractions to per cent and back. It is also necessary to develop the fact that 100% of something means the whole or all of it.

Usages of per cent. There are three different usages of per cent. Some writers advocate teaching two usages in the seventh grade and saving the third for the eighth grade. More recently texts advocate teaching the three "cases" in the seventh grade.

Examples of the three usages are as follows:

1. Finding a per cent of a number, as 6% of 240 = __?__
2. Finding what per cent one number is of another number, as 16 = __?__ % of 64



12 squares are shaded

88 squares are white

12/100 of the squares are shaded

12% of the squares are shaded

Figure 11.4

3. Finding the number when a per cent of it is given, as 6 = 30% of ___?___ .

Since *per cent* means per hundred, the per cent in a problem should be changed to the equivalent common or decimal fraction. To find 6% of 240 means find 6/100 of 240. This, from a study of common or decimal fractions, means 6/100 × 240 or .06 × 240.

The ratio of two numbers may be expressed as a per cent. The ratio should first be expressed as hundredths and then as per cent. For example, a student got 32 questions correct on a 40-question quiz. What per cent did he get correct?

$$\frac{32}{40} = \frac{80}{100} = 80\%$$

The third usage of per cent usually causes students some difficulty. Brueckner[3] noted that lack of social applications may account for students' difficulties in finding the whole amount when a part and the per cent are given. Such examples as 6 = 30% of ___?___ do not have very many applications. Again the teacher must demonstrate the meaning of "find a per cent of a number." It means multiply. Thus

$$6 = .30 \times \underline{\quad ? \quad}$$

Using the already understood relationship of a product to its factors, students may see that

$$\underline{\quad ? \quad} = 6 \div .30$$

Some writers advocate delaying the introduction of the percentage formula beyond the seventh grade. Many seventh-grade students, however, may be ready to work with this formula as a generalization of all three kinds of per cent usage. The formula is usually written

$$p = b \times r$$

where
p = percentage or part
b = base or whole
r = rate or per cent

The introduction of per cents greater than 100 per cent may come in the seventh grade. Students usually wonder how we can have more than

[3] Brueckner, *op. cit.*, pp. 239–40.

all or the whole of something. Per cents greater than 100 per cent come about when we compare two numbers or convert their ratio to a per cent. When we compare a smaller number to a larger number we get a ratio less than 1 and a per cent less than 100. When we compare a larger number to a smaller number we get a ratio more than 1 and a per cent larger than 100. Thus to compare 32 to 20 gives us a ratio of 32/20 or 8/5.

$$8/5 = 160/100 \quad \text{or} \quad 160\%$$

Then we may say that 32 is 160% of 20.

MEASUREMENT

In the first six years, students are informally introduced to many of our standard measures. Various units are applied to practical problems and students are introduced to certain denominate numbers. In the seventh grade the approximate nature of measurements may be emphasized. This may be demonstrated by letting students measure the length of a stick with a ruler marked in eights. Then the distribution of the measurements may be shown. The range, median, and average of the set of measurements may be found. The distribution may be displayed on a simple graph (Fig. 11.5). Need for accuracy can be impressed with such a display.

The use of denominate numbers in computation should be understood. First, students should understand that in computation only units in the same system of measure may be added. For example, we cannot add units of length and units of liquid measure. We may add units of length to units of length and units of volume to units of volume.

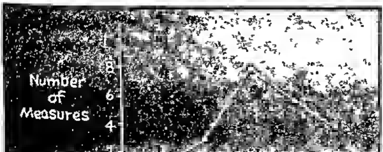


Figure 11.5

Also, within the same system of measure, such as length, we may add only coefficients of "like" units. Thus, we may add feet to feet, yards to yards, or inches to inches. We may put feet and inches together, as in the measure 4 ft 3 in., but we must change units if we are to get one numerical coefficient, such as 51 in.

INFORMAL GEOMETRY

Many basic concepts and techniques of geometry may be presented in the upper elementary grades. At the seventh-grade level, geometry is called *informal* because formal argumentations are not presented.

Uses of the compass and straightedge are extended. Further work in geometric constructions using these instruments is presented. Lines and points are accepted without definition. Line segments as parts of a line are defined and used in basic constructions. These include bisecting a line, erecting a perpendicular to a line, and drawing a parallel line through a point to a given line.

Concepts of angles and the measurement of angles by protractors are presented in the seventh grade. An *angle* is usually described as a figure formed by intersecting lines or as being formed by rotating a line in a plane about a point. Drawings and illustrations can be used to develop needed concepts. The rotation of a line concept may help students visualize the meaning of an angle since there is a tendency for students to confuse length of sides with size of an angle. Use of a clock and the rotation of its hands can be used to demonstrate what is meant by the size of an angle. Pupils may learn to construct an angle equal to a given angle and to bisect a given angle.

Geometric figures. Students should become generally aware of the geometric forms about them. They can best do this if specific forms are studied and identified as they exist in their surroundings. Teachers may have a collection of geometric forms made from various materials, wood, tile, cardboard, glass. Students may make displays with examples of geometric forms. Geometric patterns or forms are abundant in our everyday world and students may supply many items illustrating these patterns. Some geometric figures should be studied in detail. These include the circle, triangles of various kinds, quadrilaterals, the pentagon, and hexagon.

A *circle* is defined as a plane figure all of whose points are the same distance from a point called the *center*. A line segment through the center and ending on the circle is called a *diameter*. A line segment from the center to a point on the circle is called a *radius*. A radius is half as

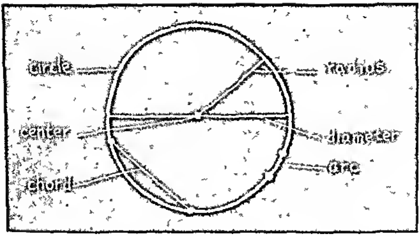
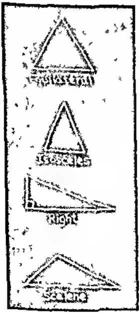


Figure 11.6

long as a diameter. The distance around a circle is called *the circumference*. A *chord* is a line segment with ends on the circle. The diameter is a chord, but all chords are not diameters. An *arc* is part of a circle. These parts may be illustrated (Fig. 11.6).

A *triangle* is a closed figure having three line segments for its sides. Special names have been given to some triangles. These names are based on the angles made by the sides of the triangles (Fig. 11.7). The sum of the angles in a triangle is 180°. The rigidity of triangles should be demonstrated. Use of this property in the construction of bridges and buildings may be studied.

Closed figures with four line segments for their sides are called *quadrilaterals*. Some special kinds of quadrilaterals may be identified (Fig. 11.8).



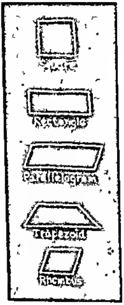
An equilateral triangle has three equal angles and three equal sides.

An isosceles triangle has two equal angles and two equal sides.

A right triangle has one 90° angle.

A scalene triangle has three unequal angles and three unequal sides.

Figure 11.7



A square has four equal sides and angles.

A rectangle has four equal angles and opposite sides equal.

A parallelogram has opposite sides parallel and opposite angles equal.

A trapezoid has one pair of parallel sides.

A rhombus has four equal sides.

Figure 11.8

Properties of these figures should be explored, including the fact that the sum of the angles in a quadrilateral is always 360°.

The word "polygon" means *many angles*. All plane closed figures having straight lines for their sides are called polygons. Regular polygons having five, six, and eight sides may be demonstrated and these constructions studied. These figures may be used in making various geometric patterns or designs.

Measurement in geometry. After students have developed some knowledge and skill in working with points, lines, angles, certain geometric figures, and relationships among these concepts, measurement may be applied to figures of various kinds. Many geometric figures are basic to our everyday life. In our homes, our roads and bridges, our automobiles, and our industries and their products we find applications of circles, polygons, and solid geometric figures of many kinds. In some cases, we are concerned with length and perimeter; in others, with volume and shape. Measurement in geometry may be of three kinds: (1) linear measure for length or distances around perimeters, (2) square measure for areas of plane figures or surfaces of solid figures, (3) cubic measure for volume of various geometric solids. All these measures have many practical applications as well as mathematical and aesthetic values.

Students at this grade level should be able to understand and use formulas. An understanding of measurement in geometry, however, comes from many concrete illustrations, written descriptions, and

definitions. Statement of a measurement fact as a formula should come after understanding has been developed.

Square measure is introduced by illustrating, describing, and defining a unit of square measure, such as the square inch. The use of the square inch in measuring area in a rectangle then may be considered. Many examples should be used. Visual materials are available to demonstrate area of rectangles. The rule comes last and after understanding. Students may then write or "discover" the rule for themselves. On such a foundation, an understanding of area of circles, various polygons, and surfaces may be developed and rules or formulas stated. Formulas for perimeter, area, and volume of common geometric figures are given in Chapter 13.

SETS AND NUMBERS

Just when sets and operations on sets should be introduced in arithmetic has not been settled. Some textbooks introduce sets in the seventh grade and extend the concepts in the eighth grade. Suppes[4] has conducted trial programs which introduce some concepts of sets in grade one. In any case, the importance of developing better understanding of modern mathematical ideas justifies some work on sets, their relation to numbers, number statements, and variables in grades seven and eight.

Sets should be presented simply as collections of things. The students in your class, the students' desks, the boys in your class, the girls in your class, the persons with blue eyes in your school—all are sets of things. Sets do not have to be physically collected together in order to be defined. The set of all National League baseball players may be widely separated in space and time, yet it is a definite set. The names of sets are included within braces { } to denote a set. For example: {John, Mary, Jane, Bill} is a set of children. The "things" may or may not be physical objects.

Sets of things are themselves things. Therefore, we may have sets of sets. Your class is a set of pupils. Your school is a set of classes Members of sets are called *elements of the set.* The symbol "$\in$" means "is a member of." Thus, if

R is the set {John, Mary, Jane, Bill}

[4] Patrick Suppes and Blair A. McKnight, "Sets and Numbers in Grade One, 1959–60," *The Arithmetic Teacher*, 8 (October, 1961), 287–90.

then we may say that

$$\text{John} \in R$$

Pupils should join the teacher in defining and exhibiting different kinds of sets. This will help them see that sets may be defined in two different ways. One way is by tabulating or naming the members, the other is by giving a rule for deciding whether things are members of the set or not.

A set is called a *subset* of a second set if every member of the first set is also a member of the second set. For example, in the foregoing set R, the children whose names begin with the letter "J" form a subset of R. We may call this subset J. We then write $J \subset R$ which is read "J is a subset of R." Write other examples of subsets. Is R a subset of itself?

We may assign a number to a set. The number tells us how many members are in the set. Sets may be matched member by member. For example, the sets $\{1, 2, 3\}$ and $\{a, b, c\}$ may be matched as follows:

$$1 \longleftrightarrow a$$
$$2 \longleftrightarrow b$$
$$3 \longleftrightarrow c$$

The sets are said to be in a "one-to-one correspondence." Such sets are said to be *equivalent*. Equal sets have identical members. How do they differ from equivalent sets?

Venn diagrams may be used to illustrate membership in sets and subsets. Let C be the set of members in your class. Let G be the girls in your class (Fig. 11.9). Then $G \subset C$.

Let E be the set of blue-eyed students in your class. Now some girls are blue-eyed, but not all blue-eyed students are girls. We could represent this as in Fig. 11.10. Thus, $G \subset C$ and $E \subset C$. Describe the relation between G and E. Let your students make up other examples of sets and illustrate with Venn diagrams.

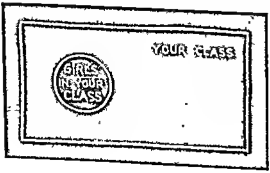


Figure 11.9

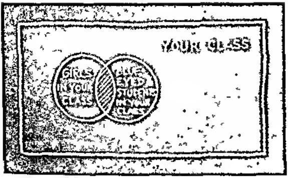


Figure 11.10

Sets may be combined to form a new set. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$. Then we may combine the two sets to get the set $\{1, 2, 3, a, b, c\}$. The new set is called the union of the two sets. The symbol $\cup$ is used to denote union. Thus

$$A \cup B = \{1, 2, 3, a, b, c\}$$

Recall that C is the set of pupils in your class. G is the set of girls in your class. *What is $C \cup G$?* Let students make up sets and form subsets and unions of sets. Illustrate with Venn diagrams.

A set with no members is called an *empty* set or *null* set. The symbol ϕ (which is the Greek letter phi) is used to denote the null set. The set of boys enrolled in a girls' school is an empty set. Most examples of empty sets seem a bit silly, but this is a very important concept. Note that an empty set is a thing. We cannot just say it is nothing. A women's reading club may be an organization of women. The members may all drop out, but the organization still exists. The individual members of a set are different from the set itself.

C. Problem Solving

Problem solving should not be considered as another topic in an already crowded arithmetic program. Instead, problem solving ought to be inherent in the development of each topic in arithmetic. What is problem solving in arithmetic? Isn't everything in arithmetic a problem? Banks[5] seems to give the simplest definition. He means by "problem" a quantitative situation described by words, requiring a quantitative answer, in which the arithmetical operations required are not provided. Thus, "If a dozen

[5] J. Houston Banks, *Learning and Teaching Arithmetic* (Boston: Allyn and Bacon, Inc., 1959), p. 364.

oranges cost \$1.08, how much does one orange cost?" is a problem, whereas "15 × 23 = ___?___" is not a problem. Marks, Purdy, and Kinney[6] take a broader view. For them, a pupil solves a problem any time a quantitative question is answered by means other than a memorized response. Others view problems in a very general sense to include not only textbook problems but problems in everyday life. Generally in arithmetic, the word "problem" refers to a word statement and a question which requires the student to make judgments about number concepts and operations and perform operations on numbers.

Difficulties in problem solving. Several reasons for difficulty in solving arithmetic problems have been identified. These include (1) reading difficulties, (2) poor skill in computation, (3) inability to relate arithmetic concepts to elements of problems. Most other difficulties will be refinements of these three. Comprehension in reading is necessary in problem solving. Arithmetic problems are likely to contain special words or terms which must be understood if the problems are to be solved. Students must develop a good reading vocabulary, understanding of the social setting for the problem, and understanding of arithmetical symbols and terms.

In order to choose number operations to be performed in a problem, pupils must relate number operations to the problem situations described in words. A simple example: Father is planning a 400-mile trip. His car averages 16 miles per gallon of gasoline on a trip. If he starts with a full tank of 20 gallons, will it be necessary for him to buy more gasoline before reaching his destination? Pupils must understand that

miles per gallon *x* number of gallons in tank
= distance in miles without refueling

and that

distance in miles without refueling must be
compared to distance of trip

The student must choose number operations to allow him to make the comparison called for in the problem. Is there only one way to approach this problem? After relationships are understood and operations are selected, obviously computational skill is needed in problem solving.

[6] John L. Marks, C. Richard Purdy, and Lucien B. Kinney, *Teaching Arithmetic for Understanding* (New York: McGraw-Hill Book Company, Inc., 1958), p. 310.

Developing problem-solving ability. What can the teacher do to help students develop ability to solve problems? Several writers[7] have noted that problem solving cannot be taught as a skill. Problem solving is inherent in understanding mathematics. The student who is proficient in producing correct answers to arithmetic examples, but cannot solve problems, may be compared to the student who is able to give sounds to syllables according to rules, but does not know the meaning of the words he sounds. You may well say that these things go together. Yes, they should, but they do not develop together in every child and for every teacher.

Efforts have been made to reduce problem solving to a series of steps. Some junior high school mathematics textbooks list steps students should follow to solve problems. Other writers[8] have listed "rules" for use in solving arithmetic problems. Most statements of steps or rules for problem solving include such items as: read the problem carefully; decide what is given; decide what is to be found; think through the steps necessary to answer the question; select and do computations; estimate answer by using simplified data; check answer carefully. There seems to be general agreement, however, that problem solving cannot be reduced to a fixed set of steps or rules. Instead, lists of steps should be considered only generally descriptive of what a person does when he solves a problem.

If arithmetic is understood as a structure of related concepts, then problem solving for the student becomes a matter of recognizing patterns and fitting what is given in a problem to a pattern already understood. For example, how frequently do we find the relationship between divisor, dividend, and quotient appearing in arithmetic problems? Students should know that divisor times quotient equals dividend and dividend divided by quotient equals divisor. The "three cases" of this relationship are found for whole numbers, for common fractions, for decimal fractions, and for per cent. Understanding of this relationship is needed for problems on topics ranging from interest and profit to perimeters and areas of geometrical figures. Once the full mathematical meaning of the equation $a \times b = c$ is developed for the several kinds of numbers, a whole range of problems may be solved. How many formulas or types of problems can you think of which are

[7] Leo J. Brueckner, Foster E. Grossnickle, and John Reckzeh, *Developing Mathematical Understandings* (New York: Holt, Rinehart & Winston, Inc., 1957), p. 310.

[8] Wilbur H. Dutton and L. J. Adams, *Arithmetic for Teachers* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1961), pp. 178–80.

based on this simple relation?

If $$a \times b = c$$

then $$a = \frac{c}{b}$$

and $$b = \frac{c}{a}$$

Thus, teachers should strive for generalizations along with teaching specific meanings or techniques of arithmetic.

Procedures or techniques for helping students improve in problem solving may be summarized:

1. At each point help students to generalize or see a pattern when a specific concept or operation is taught. Do this by showing application of the concept or operation to a variety of concrete situations.
2. Teach reading skills needed in arithmetic.[9]
3. Help students develop techniques for making problems more concrete, such as
   a. Using simplified numbers in the problem
   b. Drawing a picture or diagram of the problem
   c. Rewording by student
   d. Analyzing problems by use of rules or steps
   e. Use concrete materials related to social situation described in the problem
   f. Provide first-hand experience, such as a trip to a bank when studying interest or a store when studying profit, list price, or discount
4. Develop other devices for approaching problem solving, such as
   a. Making up problems
   b. Stating a problem without numbers and verbalizing what is given and what is asked
   c. Reasoning from what is required back to what is given
   d. Relating a problem to a simple analogous one
   e. Looking for hidden questions
   f. Estimating or checking reasonableness of answers

[9] Brueckner, *op. cit.*, pp. 314–19.

D. The Exceptional Learner

Many studies have shown that students entering the junior high grades vary widely in achievement level and ability to do academic work. They vary widely both in arithmetic achievement and in ability to learn arithmetic. Student differences in arithmetic in the seventh grade may be due to a number of factors. These factors include (1) innate mental potential, (2) home background and environment, (3) maturity level of student, (4) general academic achievement level, (5) attitudes toward arithmetic, (6) previous teaching student has received.

Many different kinds of administrative arrangements are now being made to care for individual differences in the junior high school. These range from differentiated assignments and grouping within the classroom to honors and remedial classes in separate classrooms.

THE SLOW LEARNER

We may recognize two kinds of slow learners. Most unselected classes include children below average in their capacity to learn. Generally the same materials used in regular instruction are used for these children. Their learning is characterized by the need for more concrete learning experiences extending over a longer period of time. The teacher needs to present very specific topics and procedures. We also will find the child who seems to have average or above average ability to learn but who is low in arithmetic achievement. Efforts to determine why these children have not achieved to capacity should be made. One thing seems clear. The teacher must accept the slow learner at the level which he has attained and attempt to improve his achievement from that point. Grossnickle and Brueckner[10] have made some excellent suggestions for working with the slow learner.

THE RAPID LEARNER

The rapid learner is characterized, not only by his ability to grasp factual material, but by his insight into meanings and ability to generalize. Often the rapid learner has simply been given more and harder arithmetic problems to solve. Since he does not need the drill or repetition that a slower learner may need to fix learning, it is likely

[10] Foster E. Grossnickle and Leo J. Brueckner, *Discovering Meanings in Arithmetic* (New York: Holt, Rinehart & Winston, Inc., 1959), pp. 394–95.

that boredom and dislike of arithmetic can result. The rather strict topical placement of arithmetic in the elementary grades along with our lock-step method of proceeding by grades, one per year, have made it difficult to provide proper experiences for the rapid learner. As a result, some kind of enrichment has come to be widely used. In practice, this has often resulted in keeping the same text and topical placement and adding some new topics, tricks, or games. This practice is not likely to extend and develop mathematical understandings in the superior student. Instead, a definite program for the development of additional mathematical concepts and their application is needed. New texts and materials are becoming available to help teachers with this problem.

Where it is not administratively possible to provide for acceleration or for a modernized program involving additional mathematical concepts, the teacher may have to use some kind of enrichment. Suggestions for enrichment may be found in several sources. Enrichment may include (1) history of arithmetic, (2) games and puzzles, (3) different algorisms, (4) new applications for arithmetic, (5) investigating some topics in depth, (6) making mathematical displays, (7) studying new topics, such as elementary statistics, probability, and set theory, (8) number systems with other bases.

Something to Think About

1. How is the role of social applications changing in upper elementary grade mathematics?

2. Why must operations with whole numbers be considered in seventh-grade mathematics?

3. Review several seventh-grade arithmetic texts, both old and new, to evaluate the degree to which newer concepts are being incorporated into the textbooks.

4. Why is per cent (that is, per hundred) more widely used than per thousand, per million, or per ten? Can you think of situations in which each of the foregoing is used?

5. Some of the concepts now featured in seventh-grade mathematics formerly were introduced in the tenth grade. How do you account for this? Can you find anything in the literature to support your answer?

6. Is there any place for concrete materials in seventh-grade arithmetic? Explain.

7. What does a teacher do with, or for, a seventh-grader who cannot add? Refer to the literature if you need to do so.

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Enrichment for particularly able students.

Mathematics in Grade Eight

CHAPTER TWELVE / MATHEMATICS IN GRADE EIGHT

Some teachers consider eighth grade to be a rather difficult year. For example, in certain school organizations, it is an elementary grade. In others, it is a junior high grade; in a 6–6 organization, it is neither of these, yet not quite in the high school. The trend, however, seems to be toward a junior high organization, with a corresponding decline in the number of 8–4 units.

Sociological changes have contributed to the confusion which arithmetic teachers sometimes experience in dealing with the eighth grade. For example, a few decades ago, most students ended their formal education after they finished "grammar school." Consequently, in arithmetic classes, considerable emphasis was put on business practices just because this was the last chance to give students this type of experience. Further, the tendency of our society to shift from rural to urban has rendered obsolete many of the "practical" problems about farming that were once prominent in the eighth-grade arithmetic program.

This chapter includes the following topics:

A. An Eighth-grade Program

B. The Exceptional Learner

A. An Eighth-grade Program

Just as there is lack of unanimity as to the general goals in eighth grade, so is there a wide degree of diversity in arithmetic content. Hence, in describing a program, about all one can do is to point out certain features that are being stressed. Obviously, teachers who see eighth grade as the culmination of an elementary program use a different approach from that of teachers who think of eighth grade as an orientation for high school mathematics.

RETEACHING AND EXTENSION

From the very first work in elementary arithmetic to the end of grade eight, certain concepts are continuously retaught, since they are of vital importance. Many of the elementary texts begin work in eighth-grade arithmetic by summarizing, reteaching, and extending these concepts.

Number structure. By the time they reach the eighth grade, students have certainly heard a great deal about the structure of numbers. Many texts, however, devote some space to a reteaching of such terms as *place value* and *place holder.* Special attention is given large numbers, including billions. The analysis of numbers is usually part of this work. For example, in the number 123,771,348,193, what is the meaning of the 8? Of the 2?

An activity frequently used with the study of number structure is that of rounding numbers. For example, estimate the sum of 212,385 and 121. Only those students who have developed a relatively high degree of understanding can round off numbers in a way that is meaningful. Frequently, eighth-grade students need additional practice in rounding off.

Some eighth-grade texts are making use of exponents in writing numbers. The students usually begin with such familiar procedures as writing $100 = 10^2$. This concept is expanded to $1000 = 10^3$ and may go considerably higher. Little attention, however, is usually given to the use of the exponential form in carrying out number operations.

Some eighth-grade arithmetic texts now present some work on systems of numerals with bases other than 10. Exercises are usually restricted to changing written numbers from one base to another and doing a few simple operations. For example: How do we write 42 (base 10) in a base-5 system of numerals? We may let xyz be the base-5 numeral for 42 (base 10). Then the z tells how many ones, the y tells

how many fives (the base), and the x tells how many twenty-fives (the base squared). Since 42 (base 10) contains 1 twenty-five, the x is 1. The 17 which remains contains 3 fives, so the y is 3. All that remain are the 2 ones, so the base 5 numeral is 132.

Although many texts and teachers' manuals do a masterful job in explaining numerals with other bases, it is vital that the teacher be clear on it before she introduces it to her class. Careful preparation is essential at this point.

Some teachers are questioning the value of having students work with number bases other than 10, especially on the ground that this does not meet the social criterion. Basically, the motive in such work is to help the student reach a better understanding of a base-10 system. Considerable experimentation will be necessary before we arrive at any conclusions about the effectiveness of this method of expanding the student's understanding of number.

Operations with whole numbers. One who has not taught might think that the addition, subtraction, multiplication, and division of whole numbers has been mastered by eighth grade. Indeed, this is true of many students. On the other hand, the assumption by the teacher that such mastery has been achieved is always dangerous.

Consequently, many texts begin the eighth-grade year by a rapid reteaching of the fundamental operations with whole numbers. It should be emphasized that diagnosis ought to be one of the prime purposes of the teacher. Although eighth-grade students have had extensive contact with the operations applied to whole numbers, some students may still be making certain errors because of misconceptions that were acquired years earlier. During this phase of reteaching, the teacher should be especially observant for patterns of errors, poor work habits, lack of self-confidence, and any other characteristics that would prevent normal progress.

Some teachers have found that they can use the reteaching of whole-number operations to present certain basic mathematical principles. For example, the commutative principle of addition is frequently illustrated. This principle says the order of adding two numbers does not change the sum ($3 + 4 = 7; 4 + 3 = 7$). *Would this principle apply to subtraction? Give the reason for your answer.* Along with this, the associative principle is usually presented. This says that numbers may be grouped in different ways to find a sum, or $(3 + 4) + 5 = 3 + (4 + 5)$.

Fractions. Generally, the pattern that has been described in the reteaching of whole numbers is also used in the reteaching of operations with both common and decimal fractions. The basic concepts of fractions are restudied. Terminology is covered as rapidly as is feasible. This leads to a variety of applications. In general, however, very little is presented in the way of new concepts or operations.

Here again, it is of vital importance that the teacher maintain a diagnostic point of view. Few areas of study offer more opportunities for misconceptions than that of common fractions. Such errors are frequently rather easy to correct. This, however, can occur *only* if the teacher becomes aware of the existence of the difficulty.

Per cent. Many beginning teachers are surprised when they learn that the per cent concept is a source of considerable confusion. This system of writing fractional numbers, however, is used in a wide variety of applications. Probably its versatility contributes to the difficulties students encounter in working with per cent.

Usually, teachers review with their eighth-grade classes the basic meaning of "per cent" as "per hundred." For example, if on a hundred board we hang 97 blue tags and 3 red tags, the red tags are present at the rate of 3 per hundred. This can also be written as $\frac{3}{100}$, or .03, or 3%. The idea of "at the rate of" is given considerable emphasis, since it is basic to an understanding.

Although the pattern of presentation varies considerably, there are certain phases of work with per cent that usually are studied, or restudied, in eighth grade. One such topic is that of expressing the same quantity in per cent, common fraction, and decimal fraction form. This involves such exercises as $\frac{3}{4} = .75 = 75\%$. True, such conversions ultimately become somewhat mechanical, but it is important that students understand, and possibly demonstrate concretely, what they are doing. Many teachers expect their students to learn several of the more common per cent equivalents, such as $\frac{1}{2} = 50\%$; $\frac{1}{3} = 33\frac{1}{3}\%$; $\frac{2}{3} = 66\frac{2}{3}\%$; $\frac{1}{4} = 25\%$; and $\frac{3}{4} = 75\%$. Some others expand this list to include sixths and eighths. Fractions written with sevenths or ninths as denominators are frequently omitted, since they are seldom used in practical situations.

A second topic that is given considerable attention is that of finding a per cent of a number. Although the topic is normally introduced with familiar numbers, such as "find 25% of \$50," there will usually be rapid progress toward such cases as "find 37.5% of \$117." *Can you explain why we point off three places in the product?* It is readily apparent

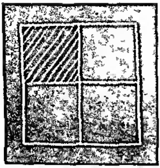

Figure 12.1

when the 37.5% is written as .375. Fractional per cents (find $\frac{1}{2}$% of 500) are usually included here.

The use of per cent as a means of expressing ratios is usually retaught in eighth grade. Frequently, this is combined with a study of simple geometric figures (Fig. 12.1). For example, the student may see that $\frac{1}{4}$ or 25% of the figure is shaded. The concept of ratio can also be applied in such areas as won–lost records in sports. If a basketball team won 20 games out of 30, the ratio of wins to games played is 20/30, or $\frac{2}{3}$, or $66\frac{2}{3}$%.

The use of per cents to describe amount of increase or decrease is sometimes a bit confusing. For example, a \$10 sweater is marked down to \$7 during a sale. Is the decrease in price $\frac{3}{7}$ or $\frac{3}{10}$? The general principle is to compare the amount of change to the number before the change occurred, so that the price decrease would be $\frac{3}{10}$ or 30%.

One of the most difficult types of per cent problems is the one in which we are to find a number when we know a certain per cent of it. For example, 80% of a number is 60. What is the number? Several methods are available to us in working such a problem. One is to think ".80 times a number $= 60$; the number will be $60 \div .80$; hence the number will be 75." Another approach would be to write the 80% as $\frac{4}{5}$. Then we would say, "$\frac{4}{5}$ of a number $= 60$; the number $= 60 \div \frac{4}{5} = 60 \times \frac{5}{4} = 75$." Another might be "If .80 of the number is 60, then .01 of it will be $\frac{60}{80} = \frac{3}{4}$. Then $\frac{100}{100}$ of it will be $\frac{3}{4} \times 100 = \frac{300}{4} = 75$." Note that the essential differences among these methods lie in the pattern of thinking; the computations involved are essentially the same.

The work with per cent in eighth-grade arithmetic usually places considerable emphasis on problem solving. The teacher should try to use real situations for problem work: sports records, ticket sales, advertisements from the local paper, and others. Many texts get into simple business practices in working with per cents. Such terms as profit, loss, discount, margin, mark-up, and overhead are frequently

introduced and used in problem work. These, however, are not emphasized as much as was the case several decades ago.

MEASUREMENT

In the earlier grades, students concentrated most of their attention on the English system of measurement, since this is the system that is accepted as the standard in America. This means that the students have worked extensively with the foot, along with its multiples and parts, and with pounds, tons, and ounces. In the upper elementary grades, however, students begin to come into contact with the metric system, particularly in science and mathematics classes.

In eighth-grade mathematics, considerable time is devoted to the metric system. Generally, the meter as a unit of length is studied first. This leads into the units that are larger (kilometer) and smaller (centimeter and others) than the meter. The simplicity and logic of this system of measures is apparent to the student after he has had experience in converting from one metric measure to others.

From the metric units of length, most texts go into other measures in this system. Frequently this includes such volume measures as the milliliter and liter. It is not unusual to have some space devoted to "weight" measures, such as the gram, kilogram, and metric ton.

Usually, however, it is necessary to introduce a confusing by-product of the study of the metric system—that of changing from metric to English measures or the opposite. It may have confused the student in earlier years that there were 12 inches in a foot and 3 feet in a yard. It will doubtless be even more puzzling to him to learn that a meter is 39.37 inches or that a pound is 453.6 grams. Although the study of these conversion factors is essential if the student, presumably with a background in English measures, is to have any conception of metric units, it is of doubtful value to place emphasis upon conversions.

Frequently, as an outgrowth of the study of metric measures, students will raise such questions as: How extensively is this system used? What parts of the world seem to use it most? Why don't we use it? Such questions make good topics for individual class reports, since most of the answers can be found in standard reference sources.

Instruments of various kinds have considerable usage in eighth-grade measurement study. For example, in working on length measures, students make use of foot rules, yardsticks, meter sticks, steel tapes, and possibly, a micrometer. Weight measures frequently involve

standard scales, spring scales, and balances. Other types of measures also involve using instruments. The use of these instruments should be thought of as more than concrete teaching. For one thing, we should expect students to develop some facility in the use of these instruments, thereby achieving a higher degree of accuracy in measures. Also, using these instruments gives opportunity to bring out the basic principle that all measures are approximations. This point is frequently confusing to students. By working with measuring instruments, however, they demonstrate that they frequently disagree with themselves when the same object is measured several different times. As instruments are improved, and as skill in the use of instruments increases, our measures become better approximations. An exact measure, however, one that cannot be improved, has not been achieved.

Many eighth-grade texts give some work in *indirect measurement*. One of these is somewhat traditional in that we measure the length of a shadow cast by an object of known height, then measure the shadow length for an object of unknown height. By using these data, one can compute the height of the unknown object. The school flagpole has long been a favorite object for this calculation. Although ratio and proportion do not get major emphasis in eighth-grade arithmetic, the teacher and class, in a carefully directed discussion, can usually evolve the principles that apply here. Incidentally, the time factor should not be overlooked in this type of problem. The shadows are longest in early morning and late afternoon, but the shadow lengths are changing at a maximum rate at these same times of the day. Hence, if the shadow measures of the known and unknown objects are several minutes apart, some error is inevitable.

There has been a tendency for more and more geometry to be included in junior high mathematics. Thus some programs now introduce the concept of similar triangles in seventh and eighth grade. Do you recall that in similar triangles (that is, triangles that have the same shape), the corresponding sides are in proportion? Maybe you remember it as $A/a = B/b$, where A and B are two sides of one triangle and a and b are corresponding sides of a similar triangle. Actually, this is the principle that was used in the shadow measurements described earlier. In the general form, however, this principle can be used to find distances across lakes or rivers, distances which would be all but impossible to measure directly.

A further application of indirect measurement is in the use of scale drawings. This procedure is usually introduced before eighth grade.

usually work with a convenient part of a line called a *line segment*. The interrelationships among points, lines, and planes are sometimes used.

In some cases, students are shown how to carry out certain types of operations using lines. For example, the method used in bisecting a line is frequently presented. In some programs, a method for drawing a line parallel to a given line is shown. Generally, however, lines are treated as essential tools for establishing geometrical figures. Hence, the line as such is given little attention.

Angles. Since the use of the protractor is frequently taught earlier than eighth grade, the concept of angles is usually somewhat familiar to students of this grade level. It is well to reteach the meaning of "degree," however, and to review the techniques of using a protractor in order to measure the size of an angle.

Some terminology is taught (or retaught), including "vertex of an angle," "sides or legs of an angle," and possibly a few others. From here, the texts usually go into types of angles, right, acute, obtuse, and straight angles, with practice in working with each. *Do you remember how these types of angles differ?* One of the more popular approaches to working with angles is in problems based upon the compass, since any type of angle can be illustrated by this device. Indeed, some texts are including fairly sophisticated navigational problems for use in eighth-grade mathematics classes.

Types of figures. Although several of the relatively simple geometric figures, such as squares, rectangles, and triangles, are usually introduced earlier than eighth grade, these are usually retaught. The concepts are expanded in various ways. For example, in working with triangles, the method for computing area is developed. Congruent triangles and similar triangles are frequently studied, along with certain applications of these concepts. Special types of triangles, scalene, isosceles, equilateral, acute, obtuse, and right, are frequently included. The fact that the sum of the angles in a triangle equals 180° is presented, along with certain applications and problems based upon this figure.

In further work with geometrical figures, the various programs vary considerably, both as to scope and depth of treatment. Hence, it would not be possible to describe a "typical" content for such texts. Among the topics being included, one finds triangles, rectangles, squares, trapezoids, the rhombus, the parallelogram, quadrilaterals, the pentagon, hexagon, and others. It is noted, of course, that several of the terms listed are overlapping. *Can you tell which ones are?*

Some programs take a limited list of figures and go into some detail as to computational procedures based upon them. Others present a more comprehensive list but, with the exception of a few of the more common ones, limit themselves to descriptive treatment. A few programs include some details of construction. These may include such items as circumscribing a hexagon about a circle or inscribing a square in a circle.

A next step in the geometry section would include the study of solid figures, this usually going into a computation of surfaces and volumes. Some texts give a relatively comprehensive treatment of this phase, going into the study of such solid figures as cubes, rectangular solids, cylinders, spheres, cones, and prisms. Certain specialized applications, such as the computation of board feet in lumber, are sometimes included.

Probably as important as geometry content is the treatment of these topics. The awesome spectacle of a formal geometric proof does not confront the eighth-grader. The concepts are presented in narrative style, with many illustrations. Further, the developmental approach is used to a large degree. Thus, students seldom begin with a formula which they apply in order to solve problems. Rather, basic principles are presented, then the students and teacher evolve a method. Ultimately, they summarize their findings in a formula. Certainly it would be inefficient to have students develop a method every time they need to use it. On the other hand, a student can make meaningful use of $A = \frac{1}{2}bh$ only if he knows why and how it works.

Good teaching is not likely to result from inadequate preparation. This is especially true of the geometry topics which have become a part of elementary mathematics. Yet in many cases, elementary teachers have rather limited background in geometry because their college curriculum did not include any work in that subject. In helping a class evolve a concept in geometry, it is vital that the teacher be clear on what she is doing. Consequently, in teaching this phase of elementary mathematics, it is essential that the teacher make careful, thorough daily preparation for teaching. Many experienced teachers can testify that it is embarrassing to have students "catch you off base" in such matters.

GRAPHS

Most of the common types of graphs are introduced to students before eighth grade. Consequently, although a considerable amount of

reteaching is needed, in most types, little by way of concept development remains to be done. Hence, the authors of the texts usually stress a higher degree of accuracy or a wider variety of applications than was the case in earlier grades.

There is an occasional criticism by teachers that the students have too many of the decisions made for them in graph preparation. Certainly if the student is told to "incorporate the following data" in a particular type of graph, he is being given little opportunity to decide which graph would serve best under the circumstances. There is no reason, however, for the teacher to follow textbook procedure blindly in such circumstances. Local data can be used, or textbook data can be incorporated into a variety of graph patterns. At any rate, the student should have an opportunity to exercise his judgment to a considerable degree in deciding which type of graph he wants to use.

The most widely used graphs in eighth grade are bar, circle, line, pictograph, and map graphs, such as are used in meteorology. There are numerous variations of these types, such as showing two sets of data on the same bar graph or using adjacent scales in showing comparisons.

In a few programs, students are introduced to coordinate graphs in seventh or eighth grade. This, you recall, has x and y axes and is divided into four quadrants. Although the concepts involved are probably not too difficult for the average eighth-grader, it is of little use to bring in this phase of graph work unless it serves a purpose. Since the most common use of this type of graph is in working with equations, it would be logical to study the two together. Hence, in those texts where coordinate graphs are used, they are usually studied as part of the work on equations, rather than as part of graphs. Also, the relationships with geometry are given considerable emphasis.

Certain very valuable work habits can be emphasized in working with graphs. For example, to be effective, a graph must be neat; many eighth-graders need to make further effort in this direction. Also, such matters as setting up and using an appropriate scale can give a student valuable experience in estimation. The teacher should keep these and related goals in mind, since it is easy to let the mechanics of graph making come to overshadow other teaching opportunities. Further, a teacher should be careful to assign this work at a reasonable speed. In one minute, the teacher can assign enough graph work to give the student several hours of intensive labor. This would be especially true of a student who is poorly coordinated and has trouble in the use of ruler, protractor, and other instruments.

ALGEBRA

For many years, algebra was thought of as something that happens in ninth grade. This pattern has gradually changed, however, and now the algebraic approach is introduced in the lower elementary grades. For example, many small children work with such equations as $3 + \square = 7$, or $3 + \underline{\quad ? \quad} = 7$. As long as students do not think that this is something that "Mama studied in high school," they usually handle it rather well.

The algebraic approach gets considerable emphasis in grades seven and eight, although frequently the term *algebra* is not used. Rather, attention is concentrated on equations, their uses, and the ways in which they can be interpreted. There will usually be some work in problem solving by use of equations.

Although there is no universal pattern in the presentation of algebra in eighth grade, one approach is extensively used. After a review of terms and concepts associated with equations, it is common practice to show that if we subtract the same amount from both sides of an equation, the remainders are equal. *Can you think of ways in which we could go back to the concrete in order to show this?* After the principle of equal subtractions has been established, it is used in the solution of such equations as $x + 8 = 10$. Here we subtract 8 from each side of the equation, so that $x = 2$.

Using the same kind of thinking, students may verify that we can solve equations through use of any of the fundamental processes, since we can add, subtract, multiply, or divide them. This work will be meaningless, however, unless we are sure that students are clear on the meaning of equations. *Can you see any place for a laboratory balance in presenting the equation concept?* Further, there is danger of glib verbalism, rather than understanding on such matters as "If equals are added to equals, the sums are equal."

Some parents and teachers find it confusing that students now learn to solve equations without learning to "transpose." This process ("carry all unknowns to the left and all knowns to the right, changing the sign of any quantity that crosses the equal sign") was the ultimate in mechanical manipulation. In all too many cases, understanding was neither expected nor encouraged. Consequently, the study of algebra was not greatly harmed when the process of transposition was no longer used.

It is noted, of course, that the method of equal additions, subtractions, multiplications, or divisions is a meaningful way of achieving

the same result one would get by transposition. For example, in $x - 5 = 2$, transposition would yield $x = 2 + 5 = 7$. By equal additions, we add 5 to each side of the equation. This gives $x - 5 + 5 = 2 + 5$, or $x = 7$.

The value of checking a solution by substituting the answer in the original equation is usually pointed out in eighth-grade texts. This procedure is excellent when taken seriously. In the checking of equations as in other types of checking, however, if one works from the presupposition of accuracy, little benefit comes from checking. For example, if a student solves $x - 2 = 6$ to get $x = 9$, he might be tempted to set up, by way of a check, that $9 - 2 = 6$. If checking is to be emphasized in such work, it should be treated as a meaningful process which can be most helpful in locating errors. It should not be just a term that appears in the heading of a long list of exercises, such as "Solve and Check."

SETS AND SENTENCES

Some eighth-grade programs are making use of algebraic approaches and terms based on set theory. Although the ideas are closely related to those of traditional algebra, the terminology is different. For example, an equation is referred to as a "sentence" since it expresses a complete thought written in symbols. Mathematical sentences with an unknown are called "open" sentences. The time-honored term *place holder* may be applied to the unknown in an open sentence such as $x + 2 = 5$.

Also, students are introduced to the symbols: $\neq$ (does not equal), $>$ (greater than), and $<$ (less than). We may have such inequalities as $x > 5$ or $x + 3 < 10$. In the eighth grade, the universe for the "variables" or unknown is usually restricted to the natural numbers. Sometimes equations and inequalities are collectively called *set selectors*, since they select a set of numbers that will make a true statement of the sentence.

For example, let the natural numbers be the universe from which we choose the "solution" set.

If $x + 3 = 10$, the solution set is $\{7\}$.
If $x > 5$, the solution set is $\{6, 7, 8, 9, \ldots\}$.
If $x > 10$ and $x < 13$, the solution set is $\{11, 12\}$.

Students may learn how to graph a solution set. Often this helps to clarify their ideas about open sentences and solution sets.

BUSINESS APPLICATIONS

The emphasis upon business procedures that long dominated some eighth-grade programs is gradually disappearing. This is inevitable, since the content at this grade level is steadily expanding to include more topics that formerly were introduced in high school. Obviously, something must be removed in order to make room for the newer phases of work. Further, eighth grade is far less "terminal" than it used to be, since the likelihood of student dropouts after "grammar school" is less. As a result, less attention is given to business practices at this grade level.

The sets of arithmetic texts that include books for grades one through eight frequently give more attention to business arithmetic at eighth-grade level than do the separate junior high school sets. The latter group of texts give much more attention to basic mathematics than to business practices.

Some topics included in the more traditional eighth-grade texts are assessed valuation, various types of insurance, the standard types of discount, stocks and bonds, checking and savings accounts, simple and compound interest, corporations and how they operate, customs' duty, the public debt, dividends, government costs and taxes, installment buying, wholesale and retail trade, and many others. Some of these topics are treated in considerable detail. Efforts to add reality are made, chiefly by asking the students to bring in "live" data.

Some junior high texts designed for use by the eighth grade give very little attention to this phase of arithmetic. It is not unusual to find such books omitting such terms as savings, taxes, and discounts. When these processes are included, they are commonly used as applications of per cent; that is, they are used illustratively. They are seldom taught as having major teaching merit in their own right.

Doubtless many teachers will be concerned as to the wisdom of this shift in emphasis. No general conclusions are possible, since conditions vary widely from one community to another. We are likely to see further changes in eighth-grade content, with the emphasis continuing to shift from the utilitarian to the mathematical. This seems to be the inevitable results of the holding power of the school, with fewer and fewer terminal eighth-grade students.

STATISTICS

A comparatively new topic in elementary mathematics is statistics and probability. This is receiving attention in several programs,

especially those designed specifically for junior high schools. A certain amount of background in set theory is usually assumed, since the nomenclature used in based upon this approach.

Students work from very elementary demonstrations to evolve a concept of probability. For example, "If you draw a marble from a box containing a red, a blue, and a white marble, what is the probability that you will draw a red one? A red *or* white one?" Graphs and frequency distributions are used concurrently in order to present certain types of data. The mean as a measure of central tendency is usually stressed, whereas the standard deviation is emphasized as a measure of variance.

Again, teachers may find themselves wondering whether it serves any useful purpose to teach statistics at eighth-grade level. This, of course, leads back to the basic question: How is the social criterion, or the element of practicality, functioning in the eighth-grade program? Corollary to this, to what degree *should* it function? At any rate, the illustrative and problem material that is used in presenting the work in elementary statistics is such as to add reality to this work. Hence, it probably does contribute to a students' basic understanding to give him background work in elementary statistics.

B. The Exceptional Learner

As has been mentioned, a first-grade class includes a wide range of ability. If, however, we assume that the intelligence quotients remain stable as the individuals progress through the elementary grades, it is apparent that the ability levels, as *measured by the mental ages, show an ever-increasing diversity.* Consequently, there is likely to be a wide range of abilities by the time the group reaches eighth grade (chronological age of fourteen or thereabouts). Constant effort on the part of the teacher is essential if the needs of all groups are to be met. And, incidentally, she must keep in mind that students of average ability (the largest group of all) are still present.

THE SLOW LEARNER

Unfortunately, the slow learner seldom encounters those phases of mathematics that stimulate students, such as enrichment materials. His time and effort have largely been devoted to more prosaic matters,

such as work on the basic facts. What is the duty of the eighth-grade teacher if the student still has not achieved the required degree of confidence in the multiplication or addition facts?

In such cases, one cannot evade the need for additional work on the fundamentals. It is to be hoped, however, that the teacher will minimize the use of a deadening type of drill. After all, this phase of mathematics has probably contributed about all it has to offer by the time the student has reached eighth grade. Rather, the teacher should try to give the slow learner practical problems in which he will need to use the fundamentals. Frequently, slow progress in this area is, at least in part, attributable to a lack of interest. Some teachers achieve good results by trying a more adult approach. The teacher might point out certain weaknesses, the more specific, the better, and urge the student to proceed on his own or with a minimum of teacher guidance. Many eighth-grade students respond positively when treated as adults, since this is a relatively new approach to them, and there is a real challenge involved. Further, if the use of enrichment materials is limited to the fast-learner group, there is a built-in caste system functioning. Many slow learners respond quite well to an occasional contact with the challenging materials that they see other students using. Some teachers, for example, testify that work with bases other than 10 can be just as fascinating and profitable to the slow learner as to any other segment of the class.

THE RAPID LEARNER

Working with the rapid learner actually becomes easier as he progresses through the grade sequence. As he develops a better background, new interests crop up constantly, interests which he can follow effectively. He has now had contact with several phases of mathematics beyond arithmetic, such as algebra and geometry. Frequently, with a minimum of guidance, he can explore such areas. Under these circumstances, source materials are abundant in the form of high school texts and reference books. And you can imagine the thrill an eighth-grader would experience if he worked some problems from a college text book. There is no reason why he shouldn't.

Since, currently, the rapid learner is receiving more attention, several organizations and companies are producing enrichment materials in mathematics. For example, if the adopted text incorporates little or nothing from the School Mathematics Study Group, their *Mathematics for Junior High School* would be most challenging.

The National Council of Teachers of Mathematics publishes some excellent enrichment materials. Two illustrations are Ringenberg's *A Portrait of 2* and Smith and Ginsburg's *Numbers and Numerals.* The Webster Publishing Company has a series called "Exploring Mathematics on Your Own." Some titles are *Sets, Sentences and Operations; The World of Statistics; Computing Devices; Short Cuts in Computing; Understanding Numeration Systems;* and *Fun with Mathematics.* Also, the Harper & Row enrichment series, cited previously, includes booklets for eighth grade. Some titles are "Calendar Fun," "Binary Numbers," "Prime Numbers," and "Guzintas" (as you would surmise, the last one is on division). And for the teacher who desires more complete information in this area, the National Council of Teachers of Mathematics publishes Schaaf's *Recreational Mathematics.* This is bibliographic in nature and cites several thousand sources of assistance for the teacher.

Something to Think About

1. A student asks why he has to study the metric system when he never uses it. How would you answer him?
2. What should be the role of business applications in upper elementary mathematics?
3. Describe several examples of what we mean by "generalizing relationships."
4. Does the fact that many new topics are coming into eighth-grade arithmetic mean that the earlier program was too easy? Explain.
5. Can the numeral 11021 occur in a base-2 number system? Explain.
6. Would you consider it in order to teach base 5 to a student who has trouble with base 10? Defend your position.
7. If we teach phases of algebra and geometry in grade eight, what happens in grade nine or ten, where such topics have traditionally been taught?

Selected References

Clark, John R., and Laura K. Eads. *Guiding Arithmetic Learning*. New York: Harcourt, Brace & World, Inc., 1954.
A particularly good treatment of measurement.

Johnson, Donovan A., and William H. Glenn. *Sets, Sentences, and Operations.* St. Louis: Webster Publishing Company, 1960.

Good reference for students and teachers dealing with basic set theory.

———. *Understanding Numeration Systems.* St. Louis: Webster Publishing Company, 1960.

Gives interesting work on a variety of number bases, with emphasis on 2, 5, *and* 12.

Two other booklets in this series, *Computing Devices* and *The World of Statistics*, would be of interest to some eight-grade students.

Marks, John L., C. Richard Purdy, and Lucien B. Kinney. *Teaching Arithmetic for Understanding.* New York: McGraw-Hill Book Company, Inc., 1958.

Good explanation of the fundamental processes, with numerous illustrations based upon the classroom.

The Mathematics Teacher. A periodical published by the National Council of Teachers of Mathematics, contains many articles of interest to junior high school mathematics teachers.

School Mathematics Study Group. *Mathematics for the Junior High School.* New Haven: Yale University Press, 1959.

A source book on various innovations in junior high school mathematics.

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Number Applications for the Teacher

We are now witnessing a revolution in the teaching of mathematics. Emphasis has changed from mere facility in computing or manipulating according to rules to understanding the rules plus facility in applying them. Historically, much of man's difficulty with mathematics has been his inability to make number symbols and operations identical with the world he perceives. This is true although abstract number certainly arose from man's concrete use of number. Our scientific and technical age demands more from us in the use and understanding of number. Yesterday's "social utility" may not meet the needs of tomorrow's complexities. Number applications in old and new ways are needed in the elementary grades.

This chapter discusses the following topics:

A. The Number Scale

B. Measurement

C. Elementary Algebra

D. Elementary Geometry

E. Probability and Statistics

F. Systems of Numeration with Other Bases

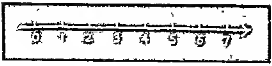


Figure 13.1

A. The Number Scale

We shall use the terms *point, straight line,* and *length* without definition. Some intuitive idea of a point and a line certainly develops early in a child. It is possible for us to represent numbers as points on a line. We choose some position on a line segment as our starting point and label it zero. We then choose some convenient length and lay off intervals from zero. The points thus located are labeled 1, 2, 3, 4, . . . , our natural numbers (Fig. 13.1). Note that we marked off the points labeled by the natural numbers to the right of the zero point, or origin. Although only a small part of a number line may be illustrated, note that we have an infinite sequence of points in a one-to-one correspondence with our natural numbers. If the length of the interval from 0 to 1 is taken as the unit of length, then each number tells its distance from 0 in terms of this unit.

The integers were developed as an extension of the natural numbers. In order to represent the integers (Fig. 13.2), simply extend the number line indefinitely either side of the origin. The positive direction is to the right, the negative direction to the left.

NUMBER OPERATIONS

Number operations have been defined for each of the number systems. Our definitions have been rather abstract. The number line may be used to demonstrate number operations, especially with the natural numbers and integers. Suppose we wish to add the natural numbers a and b. We measure out a units from 0. From the point a we measure out b units. The point (Fig. 13.3) so located is the length $a + b$ from 0.

The number line may play an important role in illustrating operations on integers. Our rules for operations on positive and negative integers were rather abstract. Applications or concrete illustrations can help in clarifying our ideas about negative integers.

We may explain the addition of integers on the number line by

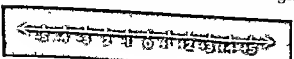


Figure 13.2

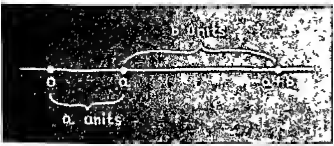


Figure 13.3

remembering that the positive direction is to the right, the negative to the left. To add a positive number, we move that many units to the right. To add a negative number, we move that many units to the left. Thus $(+4) + (+2)$ means that we start at $+4$ and move 2 units to the right; $(+4) + (-2)$ means that we start at $+4$ and move 2 units to the left (Fig. 13.4).

Subtraction may be demonstrated on the number line by remembering that subtraction is the inverse of addition. Thus $(+3) - (+2)$ means $+3 = (+2) + \underline{\quad ? \quad}$, and $(+3) - (-2)$ means $+3 = (-2) + \underline{\quad ? \quad}$. Thus, we start at $+2$ and move 1 unit right to $+3$, or we start at -2 and move 5 units right to $+3$, indicating that $(+3) - (+2) = +1$ and $(+3) - (-2) = +5$ (Fig. 13.5).

Show by drawings that to subtract a "signed" number you may change the sign and add.

RATIONAL AND REAL POINTS

We have seen how the integers may be represented by points spaced along a line. Each point is separated from the point on either side. We know which is the next point. Such a set of points or numbers is called a *discrete* set. On the number line there is a gap or space between

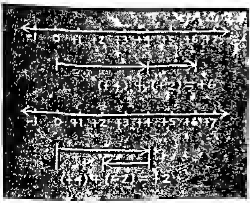


Figure 13.4

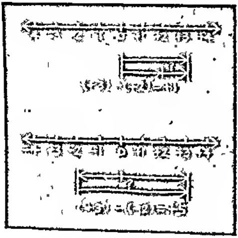


Figure 13.5

a point and the next point. The rational numbers may also be represented by points on a line. Since rational points are ordered, we have an ordered set of points. This means that, given any two rational points, we can tell which one is the greater distance from zero, or the origin. Thus, $\frac{7}{8}$ is further from zero than is $\frac{1}{2}$. Now, we may note that between any two points representing rational numbers there is always another point (Fig. 13.6). Thus, there is an infinite set of points between zero and 1. The gaps seem to be filled. Such a set is said to be *dense*.

We know that the rational numbers are isomorphic to a subset of the real numbers. Thus, there are irrational numbers "between" the rationals even though the rationals form a dense set. For example, the $\sqrt{2}$ cannot be represented as a rational, but we know it lies somewhere between $1\frac{4}{10}$ and $1\frac{5}{10}$. On the number line we may illustrate $\sqrt{2}$. The real numbers complete the number line (Fig. 13.7) because every point on the line corresponds to some real number, and for every real number there is a point on the number line.

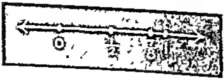


Figure 13.6

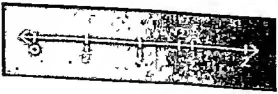


Figure 13.7

Figure 13.8

Figure 13.9

Figure 13.10

GRAPHS AND THE NUMBER LINE

The number line may be used to graph or give a visual illustration of the solution set of a mathematical statement. Some examples follow:

In the domain of natural numbers, what is the solution set for $x < 5$? The large dots on the line (Fig. 13.8) show the solution set.

In the domain of the integers, what is the solution set (Fig. 13.9) of $-3 < x < +3$?

In the domain of the nonnegative real numbers, what is the solution set for $x + 1 \geq 4$? The solution set (Fig. 13.10) consists of all real numbers larger than 3 and 3 itself. The heavy line indicates that all real numbers equal to, and larger than, 3 make up the solution set.

Note that a number and its negative are the same distance from the origin. This distance is called the *absolute* value of the numbers. It is symbolized by placing the number between vertical bars. Note that

$$|+3| = |-3| = 3$$

"3" means the same as "+3." A numeral with no sign is understood to represent a positive number.

B. Measurement

Various forms of measurement perhaps make up man's most frequent uses of number. In fact, hardly any phase of our life is without measurement of some sort. We have seen how the idea of one-to-one correspondence between sets can be used to generate a basic concept of number, the cardinal or natural numbers. This idea could be used only when we had sets composed of discrete

elements. Early man probably used some kind of matching process to satisfy his need for measuring continuous properties, such as length, weight, or volume. These were properties not composed of discrete elements which could be counted. What, then, is the nature of measurement?

MEASURES AND UNITS

Basically, measurement is a comparison process. Some unit is arbitrarily selected for the property to be measured. The *measure* of the property in some particular case is a number telling how many units of the property are present in the case under consideration. For example, we use the *foot* as a common unit to measure length. Using a rule marked in feet, we compare it with some object whose length we wish to measure. The number telling how many feet of length we find is called the *measure* of the object. All measurements are approximations whose accuracy depends upon the instrument we use to determine the measure.

In nearly all measurement, we have a known unit and apply it to some property to determine how many units are contained. Estimating measures is an important part of everyday life. We need to have some first-hand experience with the standard units adopted for use. Thus, it means little to tell a child a room is 10½ feet wide if the child has had no experience with the unit *foot.*

Standard units of measurement are usually defined by law. The complexity of our industry and commerce made this necessary. The English system of weights and measures has been established by act of Congress. Its units, however, have now been defined in terms of metric units which are used in most scientific work and in other countries in commerce.

ADDITIVE MEASURES

Most of our commonly used measures are *additive.* This means that if we have measures for two parts, then these measures added gives us the measure of the two parts joined or put together. Measures of length, weight, time, volume, and area may be additive. Some measures are not additive. The intensity of sound is measured in decibels. Yet, ordinarily, two 10-decibel sources of sound heard together do not give 20 decibels of sound. Two 6-volt batteries connected in series provide 12 volts, but connected in parallel they provide

6 volts. Most measures of concern to elementary-grade students will be additive, but some caution should be exercised in computation with measures.

COMPUTATIONS WITH MEASURES

Since measurement is a comparison with some kind of standard unit, it necessarily involves some error. If we are to measure the width of a room with a 6-foot carpenter's rule, we must lay the rule down, mark the end, place the rule on this mark, and so on. We can never be precise about our marks and our laying down of the rule. Even the measuring of liquids in a laboratory cannot be exact. A liter measure has a fine line indicating the level of the liquid to give 1 liter. But the surface of a liquid is not perfectly flat. What point on the curved surface do you make even with the line? Examples of weight, time, or other properties could be cited to indicate that all measures are approximations.

Accuracy in measurement. Precision in measurement is limited by the instruments we use and by our own limitations. Most carpenters' rules are marked off in sixteenths of an inch. This means the user, by being very careful, can measure a length to the nearest sixteenth of an inch. Some carpenters' rules may be marked off in thirty-seconds and even sixty-fourths of an inch, but few persons can distinguish such markings in making a measurement. Thus, the precision of the measuring instrument and our own physical limitations limit preciseness in measurement. If our rule is marked off in sixteenths of an inch, then we must make a judgment as to the nearest sixteenth. Suppose we are measuring the width of a board. The width appears to fall somewhere between $5\frac{10}{16}$ and $5\frac{11}{16}$. We must decide which mark on the rule comes closer to the edge of the board. If the edge seems to fall about halfway between the marks, two persons may decide differently and thus report two different measures. The smallest unit used in a measure determines its precision.

Accuracy in measurement refers to the ratio of the possible error to the total error. Suppose our board measured $5\frac{11}{16}$ inches. This means the measure was nearer $5\frac{11}{16}$ than $5\frac{10}{16}$ or $5\frac{12}{16}$; that is, the measure was between $5\frac{21}{32}$ and $5\frac{23}{32}$. The true width of the board could differ from our measure by, at most, $\frac{1}{32}$ inch. The accuracy is $\frac{1}{32} \div 5\frac{11}{16}$. The accuracy is $\frac{1}{182}$:

$$\frac{\frac{1}{32}}{5\frac{11}{16}} = \frac{\frac{1}{32}}{\frac{91}{16}} = \frac{1}{182}$$

We often read that the distance to the sun is 93,000,000 miles to the nearest 1 million miles. This is an accuracy of $\frac{1}{186}$. Which is the more accurate—our measure of the width of the board or the measure of the distance to the sun? Note that we may compare accuracy of different kinds of measures, but not preciseness.

Approximations. Irrational numbers may be expressed as infinite decimals. They may be written with whatever degree of accuracy we wish. Thus, we may write $\sqrt{2} = 1.41$ and $\pi = 3.14$. The rational $\frac{1}{3}$ may be written .33. Each of these numbers is an approximation. Measures are also approximations. When decimal notation is used for measures, preciseness is usually indicated by the last significant digit. Thus, 5.6 centimeters is precise to the nearest tenth of a centimeter, whereas 5.63 centimeters is precise to the nearest hundredth of a centimeter. Sometimes precision is written by indicating a correction. For example, $5.6 \pm .02$ centimeters means that the measure lies between 5.58 and 5.62 centimeters.

A concept of significant digits is important. Each of the digits 1 to 9 is always significant. Zero is significant if it is between two other digits (1–9). All digits in the following numbers are significant: .304, 570.4, 3.006, 70.03. A zero is not significant if it is to the left of all other digits (1–9). Zeros in the following numbers are not significant: .0067, 0.06, .0000015. Any zero to the right of all other digits is considered significant unless otherwise stated.

Computations with measures. Frequently, in computation with measures, we get digits which make a result seem more precise than it really is. We may wish to *round off* such results. Rounding-off is done by replacing digits by zeros in integers or whole numbers. We round off a decimal fraction by simply omitting digits. We drop or replace digits with zeros from right to left in a decimal numeral. If the last digit dropped or replaced by zero is less than 5, the last remaining digit is left as it is. If the last digit dropped or replaced by zero is more then 5, the last remaining digit is increased by 1. If the last digit dropped or replaced by zero is 5, we increase the last remaining digit only if it becomes an even digit. *Examples:*

637 rounded to nearest ten is 640.

5.43 rounded to nearest tenth is 5.4.

8.375 rounded to nearest hundredth is 8.38.

6450 rounded to nearest hundred is 6400.

We could follow a rule of increasing any last remaining digit by one if the last digit dropped or changed to zero is 5 or more.

Often we need to add approximate numbers or measures with different degrees of preciseness. The sum of measures can be no more precise than the least precise of the measures being added. Ordinarily, we follow a rule of rounding off all measures to have the same preciseness as the least precise measure before adding. *Example:* What is the total of the following weights: 136 grams, 12.5 grams, 2.63 grams?

The least precise measure is 136 grams which is expressed to the nearest gram. Rounding off, we have

136	is	136	
12.5	is	12	
2.63	is	3	
Total		151	grams

In multiplying numbers that are approximations or measures, we round off the product so that it has the same number of significant digits as there are in the number with the fewer significant digits. This is only a rough guide or rule and does not always insure that the product has the same accuracy as the least accurate of the numbers being multiplied. *Example:* A room measures 9.3 by 14.7 feet. Find its area. Multiplying we have,

$$\begin{array}{r} 14.7 \\ 9.3 \\ \hline 441 \\ 1323 \\ \hline 136.71 \end{array}$$

Compute area with maximum and minimum value of measures and show that only the first two digits on the left can be known for certain.

SYSTEMS OF MEASURES

The measures most commonly taught in the elementary grades include length, weight, capacity (liquid and dry), time, and square measure. Buckingham[1] gives a rather detailed account of how each of these measures developed in other countries and in the United States.

[1] Burdette R. Buckingham, *Elementary Arithmetic* (Boston: Ginn & Company, 1953), pp. 456–735.

The systems of common measures used in the United States are derived from those of England. For centuries, various monarchs, ruling groups, and scientific societies aided in standardizing units in England and other European countries. In the United States, the Constitution gives Congress powers to fix standards of measure. Standard units of measure have been defined by Congress, though control of measuring instruments has been left to the states. (Tables of the common units of measure in the United States are given in this chapter.)

COMMON UNITS OF MEASURE

Measures of Length

12 inches = 1 foot	5280 feet = 1 mile
3 feet = 1 yard	1760 yards = 1 mile
$5\frac{1}{2}$ yards = 1 rod	320 rods = 1 mile
6 feet = 1 fathom	6080 feet = 1 nautical mile

Measures of Weight

16 ounces = 1 pound	2000 pounds = 1 ton
100 pounds = 1 hundredweight	2240 pounds = 1 long ton

Measures of Capacity (liquid)

2 cups = 1 pint	4 quarts = 1 gallon
2 pints = 1 quart	$31\frac{1}{2}$ gallons = 1 barrel

Measures of Capacity (dry)

2 pints = 1 quart	4 pecks = 1 bushel
8 quarts = 1 peck	(Dry and liquid quarts are not equal.)

Measures of Time

60 seconds = 1 minute	28 to 31 days = 1 calendar month
60 minutes = 1 hour	365 days = 1 common year
24 hours = 1 day	366 days = 1 leap year
7 days = 1 week	100 years = 1 century

Square Measures

144 square inches = 1 square foot
9 square feet = 1 square yard
$30\frac{1}{4}$ square yards = 1 square rod
160 square rods = 1 acre
640 acres = 1 square mile (section)
36 square miles = 1 township

The metric system. The metric system has been adopted for scientific and technical use in all countries and for everyday use in most countries except the United States and England. The system was an attempt to construct units of measure related to our decimal system of numeration. The ratios between units in the metric system are in powers of 10.

Basic units in the metric system are the meter, the gram, and the liter, which are measures of length, weight, and volume. Other units are formed from these by using a standard set of prefixes. Conversion from one unit to the next unit is a matter of multiplying or dividing by 10.

The metric system relates length, weight, and volume. A cubic centimeter of water under standard conditions weighs a gram. Thus, a liter of water (1000 cubic centimeters) has a weight of 1000 grams, or a kilogram. No such simple relationship exists in the English system of measures.

megameter = 1,000,000 units	decimeter = $\frac{1}{10}$ unit
kilometer = 1,000 units	centimeter = $\frac{1}{100}$ unit
hectometer = 100 units	millimeter = $\frac{1}{1000}$ unit
decameter = 10 units	micrometer = $\frac{1}{1,000,000}$ unit
meter = basic unit	

Since we have one system of measures in everyday use and another, the metric system, for scientific measurement, occasionally there is need to convert from one system to the other. Some of the most frequently used conversion constants are as follows:

centimeter = .3937 inch	inch = 2.54 centimeters
meter = 39.37 inches	foot = .3048 meter
kilometer = .62137 mile	mile = 1.6093 kilometers
kilogram = 2.2046 pounds	ounce = 28.35 grams
liter = 1.0567 liquid quarts	pound = .4536 kilogram
	liquid quart = .9464 liter

The metric system also has a unit for land measure called the *are*, which is 100 square meters. The hectare is 100 ares, or 10,000 square meters. A square mile is 259 hectares. An acre is .4046 hectare.

There are several reasons for teaching the metric system. Its structure is like our decimal system. Working with its units may help develop insight into the structure of our system of numeration.

C. Elementary Algebra

The notion of *set* may be useful in developing elementary algebraic concepts. In fact, our use of sets in developing some basic ideas about numbers has employed simple mathematical statements or sentences. These are the building blocks for more advanced algebraic concepts.

SETS AND ALGEBRA

Sets of numbers and number pairs are frequently encountered in real-life situations. Our early experiences in counting a set of objects involves pairing the objects with the number names one, two, three, and so on. We observe number pairs daily in graphs and charts of various kinds. We have observed how useful the number-pair concept is in developing our number systems.

Mathematical statements or sentences are frequently encountered in arithmetic and simple algebra. We make judgments as to whether such statements are true or false. Such judgments are also made about statements outside the realm of mathematics. Consider the following sentence: "John is in my class at school." Suppose the word, "John," is replaced by a blank to get "_____ is in my class at school." This is called an *open sentence*, and it is neither true nor false. It becomes true or false only when a name is inserted in place of the blank. We may place some symbol, such as a letter, in place of the name. We then have the open sentence, "x is in my class at school." This letter is called a *variable*. Now, the name of any member of the class may be substituted for the variable, and we have a true sentence. Any other name would make the sentence false.

In arithmetic or simple algebra, open sentences usually take the form of equations or inequalities. *Examples:*

$$x + 5 = 12$$

$$y - 3 > 5$$

$$3z = 15$$

$$x < 4$$

The role of the letters in the preceding open sentences is that of place holder for some set of objects in the *universe* of objects being considered. The set of objects within the universe which may be substituted for the letter to give true sentences is called the *solution set*. The solution sets in the universe of natural numbers for the open sentences above are

$$\{7\}$$
$$\{9, 10, 11, \ldots\}$$
$$\{5\}$$
$$\{0, 1, 2, 3\}$$

These sentences are called *sentences in one variable*. In algebra, we may be concerned with sentences with more than one variable.

SOLVING EQUATIONS AND INEQUALITIES

The solution set for an equation may be found by trying each number in the universal set to see whether the open sentence represented by the equation becomes a true sentence. This is, of course, usually an impossible task. Finding the solution set is called *solving the equation*. Simple equations in one variable may be solved by making use of a concept of balance in an equation. Previously, we have defined the "equals" mark as indicating that two symbols or sets of symbols are different names for the same thing. On this basis, we assume the following operations will not change the balance of the equation:

1. Adding the same number to each side of the equation
2. Subtracting the same number from each side of the equation
3. Multiplying each side of the equation by the same number
4. Dividing each side of the equation by the same number. (Remember that division by zero is excluded.)

Suppose we wish to solve the equation $x + 5 = 12$. Subtract 5 from both sides of the equation.

$$\begin{array}{rcr} x + 5 & = & 12 \\ -5 & & -5 \\ \hline x & = & 7 \end{array}$$

To solve $3z = 15$, divide both sides by 3:

$$\frac{3z}{3} = \frac{15}{3}$$
$$z = 5$$

Solve the equation $5x - 4 = 26$. First, add 4 to both sides.

$$\begin{array}{rcl} 5x - 4 &=& 26 \\ +4 && +4 \\ \hline 5x &=& 30 \end{array}$$

Then, divide both sides by 5.

$$\frac{5x}{5} = \frac{30}{5}$$

$$x = 6$$

In each case we found, among the universe of integers, that integer which makes a true statement when substituted for the variable. The universe may be restricted to natural numbers or it may be the integers or the rationals. Arithmetic and simple algebra as taught in the elementary grades seldom make use of the real numbers.

The assumptions about equations do not apply to inequalities. The solution set to the simple inequalities encountered in arithmetic may be found by inspection. Sometimes graphing solution sets on the number line may be of value.

FORMULAS

Many applications of number may be represented by formulas. Several letters may be involved in a formula as well as numbers. The formula concept was used in considering percentage problems. The formula $p = rb$ means

$$\text{percentage} = \text{rate} \times \text{base}$$

In any application of a formula, we are given numbers to substitute for all the letters in the formula but one. We then have an equation in one variable, and our problem is to solve the equation. *Example:* What is 15% of 60?

$$p = r \times b$$

$$p = .15 \times 60$$

$$p = 9$$

Note that either rate or base may be missing in a given example, and we must solve for the missing variable.

Examples of formulas frequently encountered in arithmetic are

Per Cent

Percentage: $p = r \times b = rb$

Base: $b = p \div r = \frac{p}{r}$

Rate: $r = p \div b = \frac{p}{b}$

Interest

Interest: $i = p \times r \times t = prt$

Distance

Distance traveled: $d = rt$

Distance traveled by falling body: $s = \frac{1}{2} gt^2$

Temperature

Fahrenheit: $F = \frac{9}{5}C + 32$

Centigrade: $C = \frac{9}{3}(F - 32)$

Pythagorean Theorem

$$c^2 = a^2 + b^2$$

(hypotenuse^2 = sum of squares of legs of right triangle)

Intelligence

$$IQ = \frac{\text{MA (Mental Age)}}{\text{CA (Chronological Age)}}$$

D. Elementary Geometry

Geometry is taught in the elementary grades on an informal basis. Informal geometry deals with the length, area, volume, and shape of various kinds of figures. Simple constructions and measurements are used to verify statements about geometrical figures. This process contrasts with formal geometry which uses logical arguments to prove theorems about various kinds of figures.

NATURE OF GEOMETRY

The basic elements of geometry are points, lines, surfaces, and solids. These are undefined elements. Using these elements, basic assumptions called *postulates* are accepted without proof. Along with the undefined elements and postulates, *definitions* may be made. Now, statements called *theorems* may be proved using the undefined elements, the postulates, and definitions. These theorems once proved may be used to prove other theorems. The entire body of geometry is composed of its basic elements, definitions, postulates, and theorems.

Geometry may be divided into metric and nonmetric concepts. Metric concepts, as the name suggests, are concerned with measurements. Length, area, volume, size of angles are ideas involving measurement. Nonmetric concepts are concerned with points, lines, angles, curves, and various relationships that exist among them other than measures of length, area, volume, or size.

NONMETRIC CONCEPTS

The number line on which numbers correspond to points has already been described. A line may be considered as a set of points. Lines are represented as follows (Fig. 13.11). Arrows are used to indicate that the line extends indefinitely in either direction. The dot and letter A are used to label a point on the line. By a line we will always mean a straight line. A line segment may be represented as follows (Fig. 13.12). Points A and B are said to be the end points of the line segment $\overline{AB}$. Between any two points on a line there is always another point. *How is this like our definition of rational numbers?* Notice that any two points determine a line. Remember that points and lines are undefined mathematical terms and the marks on the paper simply are used to represent them.

We may think of a point lying on a line. Also, we may think of a line lying on a point. A line may be considered a one-dimensional

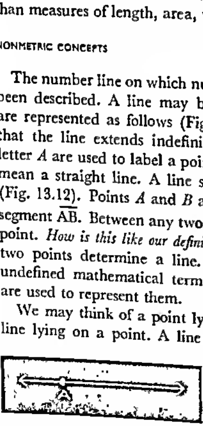


Figure 13.11

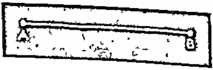


Figure 13.12

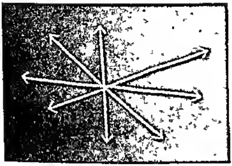


Figure 13.13

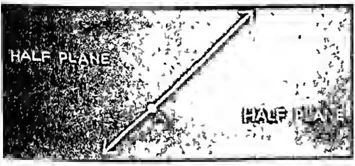


Figure 13.14

concept, but a set of lines lying on a point (Fig. 13.13) may be used to help us form some idea of a plane or two-dimensional concept. *How many lines on a point? How many points on a line?* A plane is also an undefined mathematical term. We may use some flat surface, such as a table top, to help us form some concept of a plane. Note that points and lines lie on a plane. On a plane we may have sets of points and sets of lines.

Notice that a straight line divides a plane into two half planes (Fig. 13.14).

Curves. Straight lines, broken lines, circles, and other such figures (Fig. 13.15) are called *curves.* Thus, almost any "continuous" set of points may be called a *curve.* Higher mathematics has a very precise way of defining a continuous curve which is beyond the scope of this book.

Circles, triangles, parallelograms, and figures such as the following (Fig. 13.16) are called *simple closed curves.* A simple closed curve divides

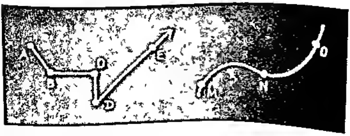


Figure 13.15

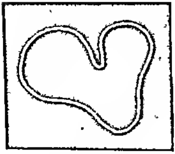


Figure 13.16

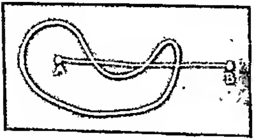


Figure 13.17

the set of points in the plane on which the figure lies into three subsets. These are the points outside the curve, those inside the curve, and those on the curve. A line segment (Fig. 13.17) determined by a point within a simple closed curve and a point without the curve has at least one point in common with the curve itself.

It is not always easy to tell whether a point is within or without a simple closed curve (Fig. 13.18). Is point A inside or outside the curve? One test is to select a point B, obviously outside the curve. Draw line segment $\overline{AB}$. If the segment *intersects* the curve an odd number of times, then point A is inside the curve. If there is an even number of intersections, then point A is outside the curve.

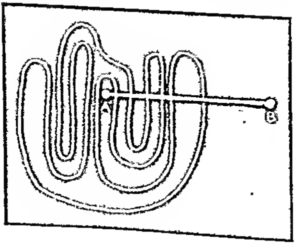


Figure 13.18

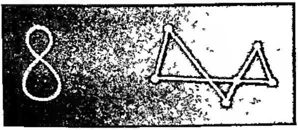


Figure 13.19

Figure 13.20

Curves, such as a figure eight (Fig. 13.19), which cross over themselves, are not simple closed curves. Nonmetric concepts have been called *projective geometry*. Also, these concepts gave rise to topology or rubber-sheet geometry where characteristics of various geometrical figures not related to distance are studied.

Angles. A half line or ray is represented in Fig. 13.20. Point 0 is said to be the *end point* of the ray. If two rays have the same end point, they are said to form an *angle* (Fig. 13.21). The point 0 is the *vertex* of the angle, and the rays are the *sides* of the angle. Point A is said to be in the *interior* of the angle.

METRIC CONCEPTS

Metric concepts are concerned with measurements. When something is to be measured, we must have a unit of measure. The measure then is a number indicating how many units are contained. A unit of measure for angles is the *degree*. Consider the following (Fig. 13.22) angles. The angle represented in (6) may be considered as an angle of 0 degrees or one of 360 degrees, written 360°. A degree, then is $\frac{1}{360}$ of the distance around a point. Names have been given to angles based on their size. Angles are measured (Fig. 13.23) by a protractor. A protractor has a vertex point and a zero point which are made to coincide with the vertex and one ray of the angle. The size of the angle is read where the other ray intersects the protractor.

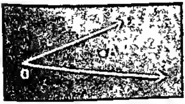


Figure 13.21

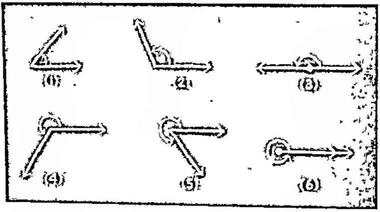


Figure 13.22

Constructions. Traditionally, the compass and straightedge have been used in making geometric constructions. Figure 13.24 shows some basic constructions which students should know; for example,

1. Bisecting a line segment
2. Bisecting an angle
3. Drawing a perpendicular to a line from a point not on the line
4. Drawing a perpendicular to a line at a point on the line
5. Drawing an angle equal to a given angle

Formulas for perimeter and area. Formulas are very useful in stating some general arithmetical principle or rule. The perimeter and area of geometrical figures may be stated as formulas. The accompanying list (Fig. 13.25) gives some commonly used formulas for perimeter and areas.

Formulas for surface and volume. Some commonly used formulas for surface and volume of three-dimensional figures are shown in Fig. 13.26.

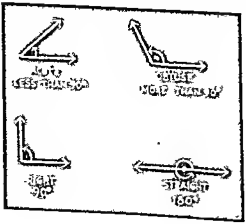


Figure 13.23

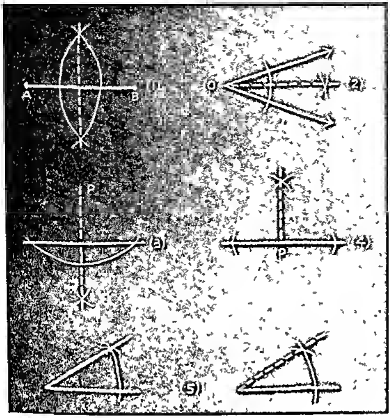
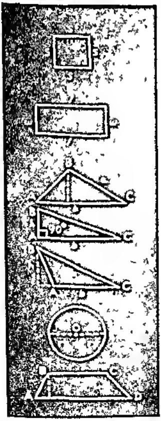


Figure 13.24



SQUARE

Perimeter: $P = s + s + s + s = 4s$

Area: $A = s \times s = s^2$

RECTANGLE

Perimeter: $P = l + w + l + w = 2l + 2w = 2(l + w)$

Area: $A = lw$

TRIANGLE

Perimeter: $P = \overline{AB} + \overline{BC} + \overline{AC} = a + b + c$

Area: $A = \frac{1}{2}bh$

CIRCLE

Circumference: $C = 2\pi r = \pi d$

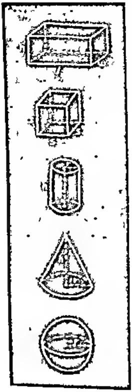
Area: $A = \pi r^2$

TRAPEZOID

Area: $A = \frac{1}{2}h(a + b)$

(Note that a trapezoid is a quadrilateral having one pair of opposite sides parallel.)

Figure 13.25



RECTANGULAR SOLID

Surface area: $S = 2wh + 2hl + 2wl$
Volume: $V = lwh$

CUBE

Surface area: $S = 6s^2$
Volume: $V = s^3$

CYLINDER

Lateral surface: $L = 2\pi rh$
Volume: $V = \pi r^2 h$

CONE

Lateral surface: $L = \pi rs$
Volume: $V = \frac{1}{3}\pi r^2 h$

SPHERE

Surface: $S = 4\pi r^2$
Volume: $V = \frac{4}{3}\pi r^3$

Figure 13.26

THEOREM OF PYTHAGORAS

One of the most famous and useful of theorems is that relating the hypotenuse to the legs of a right triangle. It may be stated as follows:

> In a right triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides (Fig. 13.27).

The simplest example, in which all three sides of a right triangle are whole numbers, is the 3-4-5 triangle. Note that

$$5^2 = 3^2 + 4^2, \quad 25 = 9 + 16$$

This theorem has been demonstrated or "proved" hundreds of ways, algebraically and geometrically.

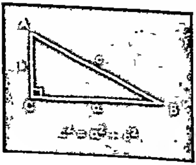


Figure 13.27

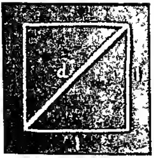


Figure 13.28

A well-known instance of failure to have whole numbers for all sides of a right triangle came in efforts to find the length of the diagonal (Fig. 13.28) of a unit square. According to the Pythagorean theorem, the diagonal d is

$$d^2 = 1^2 + 1^2$$
$$= 1 + 1 = 2,$$

or $d = \sqrt{2}$, which is an irrational number. *Show that the diagonal of any square of side s is $\sqrt{2}s$.*

E. Probability and Statistics

An important and highly useful concept is that of probability or chance. Unless we look for it, we are likely to overlook the many ways in which chance enters our everyday lives. Our weatherman predicts probable showers. Some predictions may even state that the chances of rain are 85 in 100, or 22 in 100 for tomorrow. Our insurance rates are based on probable life expectancies. A team may be given a 50-50 chance of winning.

PROBABILITY

If a certain event can happen in h ways and fail to happen in f ways, then the probability of the event happening is said to be $\frac{h}{h+f}$. The probability of the event failing to happen is $\frac{f}{h+f}$. Thus, we see that a chance or probability is a ratio or fraction which may take on values from 0 to 1. For example, suppose we have a bag containing 9 white beans and 6 black beans. We say the probability of drawing a white bean from the bag is $\frac{9}{15}$, and of drawing a black bean is $\frac{6}{15}$. Obviously, if we draw a bean from the bag it is going to be either white or black.

What, exactly, is the meaning of these ratios? The meaning may be found in the following: Suppose we draw a bean from the bag 1500 times, each time returning the bean to the bag. Then the number of white beans drawn will be very close to 900. This does not mean that the number must be exactly 900 or that it could not differ from 900 by 10, 20, or even 50.

When we toss a coin, we say the probability of getting *heads* is $\frac{1}{2}$. Again, this does not mean that if we toss the coin two times we must get a head and tail. We may get 2 heads or 2 tails. It is correct to say that it is more likely that we will get a head and tail than 2 heads or 2 tails. What, exactly, can we say about this? Suppose we toss 2 coins. We have the following possibilities:

hh, ht, th, tt

There are 4 possibilities. One of these is 2 heads, so the probability of getting 2 heads is $\frac{1}{4}$. Two of the possibilities are a head and tail (order does not matter), so the probability of getting a head and tail is $\frac{1}{2}$.

Probability began as a study of real-life events as did most mathematical topics. Probability theory, however, now may be studied as a *pure* mathematical subject not necessarily related to any actual events.

DATA AND GRAPHS

The collection and organization of data are very important parts of the problem-solving process. Attention to the organization of data into tables or graphs can be a profitable part of arithmetic instruction in the elementary grades.

Data used by pupils will usually be of two kinds—scores or measures. Scores are usually whole numbers taken from some kind of scale. Test scores, for example, may be whole numbers representing the number of correct answers on a test. Measures, such as height or weight, are approximations rounded to the accuracy desired. Both kinds of data are treated alike in elementary statistics.

In cases where there are a large number of scores or measures to be considered, *frequency distributions* and *histograms* aid in understanding the data. A table that indicates the number of scores or measures in intervals is called a *frequency distribution*. Listed on page 377 is a distribution of scores on a reading test with 50 questions.

Test Score Intervals	*Number of Pupils*
46–50	2
41–45	4
36–40	7
31–35	6
26–30	6
21–25	3
16–20	2
11–15	1
6–10	1
	$N = 32$

A graph of the frequency distribution is called a *histogram*. Test intervals are along the horizontal axis; frequency of scores in each interval, are along the vertical axis. The number shown on the horizontal axis of the graph (Fig. 13.29) is the upper limit of each interval.

CENTRAL TENDENCY

Data may be studied in other ways than reading tables or graphs. Certain mathematical concepts may help us describe important features of a set of data and allow comparisons of sets of data. One feature is called the *central tendency*. The most useful of the several

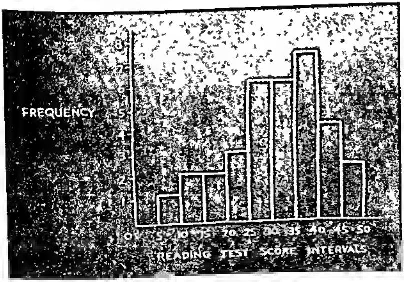


Figure 13.29

measures of central tendency is the *average*, or *arithmetic mean*. Most pupils become familiar with this measure when their test scores are "averaged" to determine a mark for a course. The *mean* is simply the sum of the scores or measures divided by the number of scores or measures. It is usually denoted by M.

$$M = \frac{\text{sum of scores}}{\text{number of scores}}$$

Suppose 15 pupils make the following scores on a 10-question quiz.

10
9
8
8
7
7
7
7
6
5
5
3
3
3
2
90

The mean is $M = \frac{90}{15} = 6$. Another measure of central tendency is called the *median*. This is the middle score in a set of scores arranged in order of size. In this example, the eighth score from the bottom is 7, which is the median score for the group. The median has some advantages in describing a set of data in that it is not affected by an extremely large or small score as is the mean. A still rougher measure of central tendency is the *mode*. This is simply the score most frequently found in a distribution. In our example, the mode is also 7.

VARIABILITY

Central tendency gives us some idea of the most frequently found score or measure about which the other scores or measures tend to cluster. Two sets of data may have the same average or mean and still differ greatly. How closely about the mean do the measures cluster? How spread out are the scores? One measure of the spread of a set of scores is the *range*. The difference between the highest and lowest scores is called the *range*. This is a very rough measure of spread, or *variability*.

A better measure of variability is called the *standard deviation*. The difference between any score in a set of scores and the mean of the set is called the *deviation* about the mean. Consider our 10-question quiz scores:

Scores	*Deviation*	*Square of Deviation*
10	4	16
9	3	9
8	2	4
8	2	4
7	1	1
7	1	1
7	1	1
7	1	1
6	0	0
5	−1	1
5	−1	1
3	−3	9
3	−3	9
3	−3	9
2	−4	16
90	0	82

Note that the scores above the mean give positive deviations, and those below the mean give negative deviations. The sum of the deviations about the mean is always zero. The last column contains the deviations squared which are all positive numbers. The average of the squared deviations is $82 \div 15 = 5.47$. Persons who are interested in statistics also use this number as a measure of the spread or variability of a set of numbers or scores. It is called the *variance* of the set of scores. The square root of the variance is called the *standard deviation*, and it is also used to measure the spread of scores. Both these measures have wide application in the study of statistics. If V stands for variance and SD for standard deviation, note that

$$V = \frac{\text{sum of squares of deviations}}{\text{number of deviations}}$$

$$SD = \sqrt{V}$$

The greater the spread, or variability, of a set of numbers, the larger their standard deviation.

F. Systems of Numeration with Other Bases

We are familiar with writing numbers in our base-10 positional system of numeration. A number such as 374 may be written as a *polynomial.*

$$374 = 3(10^2) + 7(10) + 4$$

The polynomial way of writing numbers in a positional system will help us learn to write numbers in systems of numerals with bases other than 10. Generalizations learned from these exercises may help us understand our base-10 system better.

BINARY SYSTEM

The binary system of numeration uses only the symbols 0 and 1. Objects are counted as shown in (Fig. 13.30). Note that 1 (base 2) is like 9 (base 10) in that the next numeral in the system is 10. Also, 100 follows 11 (base 2) just as 100 follows 99 (base 10).

The basic addition and multiplication facts are very simple in the binary system. Addition facts are

$$\begin{array}{r}0\\+0\\\hline 0\end{array}\qquad\begin{array}{r}0\\+1\\\hline 1\end{array}\qquad\begin{array}{r}1\\+0\\\hline 1\end{array}\qquad\begin{array}{r}1\\+1\\\hline 10\end{array}$$

Multiplication facts are

$$\begin{array}{r}0\\\times 0\\\hline 0\end{array}\qquad\begin{array}{r}0\\\times 1\\\hline 0\end{array}\qquad\begin{array}{r}1\\\times 0\\\hline 0\end{array}\qquad\begin{array}{r}1\\\times 1\\\hline 1\end{array}$$

These correspond to the 100 basic addition and 100 basic multiplication facts in the base-10 system of numerals.

The method for converting from the base-10 to the base-2 system and the reverse is rather simple. It will be demonstrated by examples. Convert 101101_2 to the base-10 numeral. It should be remembered

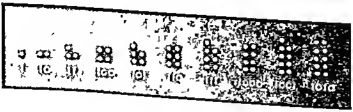


Figure 13.30

that each position in a base-2 numeral denotes a power of 2 (the base). Therefore,

$$\begin{aligned} 101101 &= 1(2^5) + 0(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1 \\ &= 2^5 + 2^3 + 2^2 + 1 \\ &= 32 + 8 + 4 + 1 \\ &= 45 \end{aligned}$$

The base-10 numeral 45 may be converted back to the equivalent base-2 numeral by *successive division* by 2.

Divide by 2	Remainder
$2\overline{)45}$, 22	1
$2\overline{)22}$, 11	0
$2\overline{)11}$, 5	1
$2\overline{)5}$, 2	1
$2\overline{)2}$, 1	0
$2\overline{)1}$, 0	1

Reading the remainders up after successive division by 2, we get the base-2 numeral 101101, which is equivalent to the base-10 numeral 45.

Addition and multiplication examples are demonstrated:

$$\begin{array}{r} 11011 \\ +1101 \\ \hline 101000 \end{array}$$

Reason as follows:

1 + 1 is the base 10; write 0 and carry ①
① + 0 + 1 is base 10; write 0 and carry ①
① + 1 + 0 is base 10; write 0 and carry ①
① + 1 + 1 may be considered in two steps:
① + 1 is base 10, 10 + 1 is 11
Write 1 and carry ①
① + 1 is 10; write 10

	Reason as follows:
11011	Write 11011
×101	Skip two places to left and write 11011 again
11011	Next, to add:
11011	Write 1
10000111	Write 1
	1 + 0 is 1; write 1
	1 + 1 is 10; write 0 and carry ①
	① + 0 + 1 is 10; write 0 and carry ①
	① + 1 is 10; write 0 and carry ①
	① + 1 is 10; write 10

THE QUINARY SYSTEM

The quinary system of numerals uses the symbols 0, 1, 2, 3, 4. The first few numerals used in counting are shown in Fig. 13.31. Note that in this system 10 comes after 4 and 100 after 44, whereas in the base-10 system 10 comes after 9 and 100 after 99. *What numeral precedes* 1000 *in the quinary system?*

Basic addition and multiplication facts are found in Fig. 13.32. As you reason through the foregoing tables, think how meaningless such a table with base-10 numerals could be to a child. Try regrouping

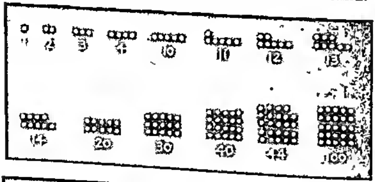


Figure 13.31

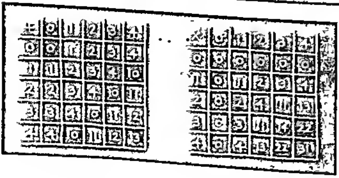


Figure 13.32

in groups of 5, and remember the polynomial way of writing numbers. For example, 3×4, we know, is 12 (in base-10 numerals) which is 2 groups of 5 and 2 more. This is 22 in the base-5 system of numerals. Suppose you did not have the base-10 system to reason in, how would you find the 22? (Group by .....'s and write in polynomial form, then write in positional form.)

Rules for converting from quinary to base-10 form and back will be demonstrated. Write 21034_5 in base-10 numerals.

$$\begin{aligned} 21034_5 &= 2(5^4) + 1(5^3) + 0(5^2) + 3(5^1) + 4 \\ &= 2(625) + 1(125) + 3(5) + 4 \\ &= 1250 + 125 + 15 + 4 \\ &= 1394 \end{aligned}$$

To convert 1394_{10} back to base-5 numerals, divide successively by 5.

Divide by 5	Remainder
5)1394	
278	4
5)278	
55	3
5)55	
11	0
5)11	
2	1
5)2	
0	2

Reading the remainders up, we get the base-5 numeral 21034 which represents the same number as the base-10 numeral 1394.

Addition and multiplication examples in base-5 numerals will be demonstrated.

	Reason as follows:
1322	$3 + 2$ is the base 10; write 0 and carry 1
+413	$1 + 1 + 2$ is 4; write 4
2240	$4 + 3$ is the base 10 and 2 more; write 2 and carry 1
	$1 + 1$ is 2; write 2

$$\begin{array}{r} 413 \\ \times 32 \\ \hline 1331 \\ 2244 \\ \hline 24321 \end{array}$$

Reason as follows:

2 × 3 is base 10 and 1 more; write 1 and carry 1
2 × 1 is 2, 2 + 1 is 3; write 3
2 × 4 is base 10 and 3 more; write 13

Next:

3 × 3 is base 10 and 4 more; write 4 and carry 1
3 × 1 is 3, 3 + 1 is 4; write 4
3 × 4 is 2 bases, or 20, and 2 more; write 22

Next, to add:

Write 1
4 + 3 is base 10 and 2 over; write 2 and carry 1
1 + 4 + 3 is base 10 and 3 over; write 3 and carry 1
1 + 2 + 1 is 4; write 4
Write 2

THE DUODECIMAL SYSTEM

Symbols used in the "dozen"-base system of numerals were illustrated in Chapter 2. Since the duodecimal system uses more symbols than the decimal system, its relation to the decimal system is different from that of the binary or quinary systems. The first few counting numerals are represented in Fig. 13.33.

Conversion of duodecimal numerals to decimal numerals, and vice versa, is illustrated by examples. Convert $2T8E5_{12}$ to equivalent decimal numerals. Remember the polynomial form. Thus, $2T8E5_{12}$ means in base-12 numerals:

$$2T8E5_{12} = 2(10^4) + T(10^3) + 8(10^2) + E(10^1) + 5$$

Now express these values in the base-10 system. Remember that T is 10, E is 11, and 10 is 12 in base-10 numerals. Therefore,

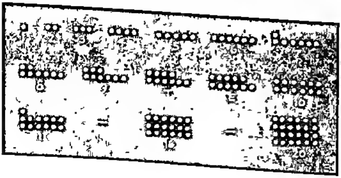


Figure 13.33

$$
\begin{aligned}
2T8E5_{12} &= 2(12^4) + 10(12^3) + 8(12^2) + 11(12) + 5 \\
&= 2(20736) + 10(1728) + 8(144) + 11(12) + 5 \\
&= 41472 + 17280 + 1152 + 132 + 5 \\
&= 60{,}041
\end{aligned}
$$

To convert back to the base-12 system, we divide successively by 12 and change the remainders to base-12 numerals where necessary.

Divide by 12	*Remainder* Base-10 Numerals	*Remainder* Base-12 Numerals
$12\overline{)60041}$		
5003	5 ⟶	5
$12\overline{)5003}$		
416	11 ⟶	E
$12\overline{)416}$		
34	8 ⟶	8
$12\overline{)34}$		
2	10 ⟶	T
$12\overline{)2}$		
0	2 ⟶	2

Reading up, we get the base-12 number $2T8E5$.

To add or multiply with duodecimal numerals meaningfully, we must regroup into dozens and units. This is exactly what we try to teach elementary school pupils to do in the decimal system. Conversion to the tens system may be done to check work. Examples of addition and multiplication will be demonstrated.

Find $8 + 7$ in the base-12 system. Re-group $8 + 7$ to get $(8 + 4) + 3 = 1$ base and 3, or 13. We write $8 + 7 = 13$. Note that although the base in the duodecimal system is also written "10," it means a dozen, or 12, in the decimal system. Note the following number facts in base-12 numerals.

$$
\begin{array}{lll}
9 + 1 = T & 8 + 2 = T & 7 + 3 = T \\
9 + 2 = E & 8 + 3 = E & 7 + 4 = E \\
9 + 3 = 10 & 8 + 4 = 10 & 7 + 5 = 10 \\
9 + 4 = 11 & 8 + 5 = 11 & 7 + 6 = 11
\end{array}
$$

$$
\begin{array}{ll}
T + 1 = E & E + 1 = 10 \\
T + 2 = 10 & E + 2 = 11 \\
T + 3 = 11 & E + 3 = 12
\end{array}
$$

If we know the basic number facts, then we may add base-12 numbers by our well-known procedure.

Reason as follows:

$$\begin{array}{r} 2T8E5 \\ +1564 \\ \hline 30259 \end{array}$$

5 + 4 is 9; write 9
6 + E is 1 base and 5, or 15; write 5 and carry 1
1 + 5 + 8 is 1 base and 2, or 12; write 2 and carry 1
1 + 1 + T is 1 base, or 10; write 0 and carry 1
1 + 2 is 3; write 3

To multiply with base-12 numerals, much regrouping is required. Try doing it without a table. You then realize why it is difficult to make the basic multiplication facts meaningful to children. The zero facts and reverse multiplication facts are not shown in the following table (Fig. 13.34). Suppose you wish to multiply 32 by 18 in base-12 numerals:

Reason as follows:

$$\begin{array}{r} 32 \\ \times 18 \\ \hline 214 \\ 32 \\ \hline 534 \end{array}$$

8 × 2 = 1 base and 4, or 14; write 4 and carry 1
8 × 3 = 2 bases, or 20, 20 + 1 = 21; write 21
1 × 2 = 2; write 2
1 × 3 = 3; write 3
Next, to add:
Write 4
2 + 1 = 3; write 3
3 + 2 = 5; write 5

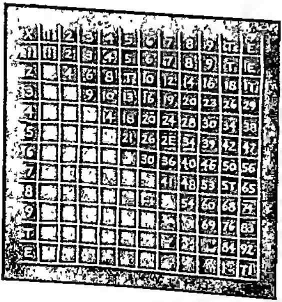


Figure 13.34

To check, change to base-10 numerals and multiply.

$$\begin{array}{lllr}
32_{12} & \longrightarrow & 3(12) + 2 = & 38 \\
\underline{18_{12}} & \longrightarrow & 1(12) + 8 = & \underline{20} \\
534 & \longrightarrow & 5(12^2) + 3(12) + 4 & 760 \\
 & & = 5(144) + 3(12) + 4 & \\
 & & = 720 + 36 + 4 & \\
 & & = 760 &
\end{array}$$

Something to Think About

1. Represent the following numbers on a number line: $6, \frac{11}{3}, \sqrt{3}, 1.66\ldots, -4, -\frac{3}{2}$, the real numbers between 2 and 3.
2. Make several mathematical statements and demonstrate the solution set of each. Include some statements with the following signs: $>$, $<$, $=$, $\neq$.
3. Why is a measurement always an approximation?
4. Demonstrate by an example the meaning of accuracy in measurement.
5. Why do we round off in computation with measures? Give an example.
6. In what sense may we state that algebra is a generalization of arithmetic?
7. What is meant by "solving" an equation? Demonstrate.
8. What is a nonmetric concept in geometry? A metric concept?
9. How may a number line be used to clarify operations on positive and negative integers?
10. Of what use is a knowledge of formulas for perimeter, area, and volume of geometric figures?
11. Find and explain one of the many ways of proving the Pythagorean theorem.
12. Collect a set of data. The data may be scores of some kind or measurements. Make a frequency distribution and histogram for the set of data. Find the mean and standard deviation.
13. Select some small numbers expressed in the base-10 system of numerals and find the base-2, base-5, and base-12 numerals for them. Do this by letting dots represent the numbers and then regrouping the dots so as to indicate the numerals in the new base.

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Changes and Inv

in Teaching

Elementary Sch

Mathematic

Changes in elementary school mathematics, like those in the secondary school, are affecting content, methods of teaching, and grade placement of topics. Generally, the grade placement of topics is being examined, and some topics are being pushed down into lower grades. There seems to be some hope of actually putting in practice the concept of teaching arithmetic "meaningfully." This is not a new concept, but it fits into plans to "*mathematize*" *arithmetic.*

This text has stressed making mathematics meaningful and understandable to children. No radical departures in arithmetic programs have been suggested. It has become apparent, however, that elementary school arithmetic teachers have been woefully unprepared for the job of teaching a modern arithmetic program. Emphasis, therefore, has been placed on some of the formal developments of arithmetic topics.

This chapter presents a brief summary of the following:

A. Changes in Elementary School Mathematics

B. Teaching Machines and Programs

C. Issues in Teaching Elementary School Mathematics

A. Changes in Elementary School Mathematics

There has been a great deal of criticism of the mathematics program in the upper elementary grades. Weaknesses noted were (1) little new material was introduced in these grades; (2) applications emphasized business and financial matters; (3) reteaching was not needed by all students; and (4) the rapid learner was not challenged. Increased use of machines to do routine computations is causing a shift of emphasis from computational skill to deeper understandings in arithmetic. In other words, social usage of number is changing. Social usage of number in the future will require much greater understanding of number and number operations.

In the process of change, we need to guard against the acceptance of new names and labels which convey no new concepts. Thus, many believe that sets and set language suddenly have been given an importance out of proportion to their contributions to the elementary school mathematics program.[1] Still others believe that Boolean algebra and formal logic have received too much attention in the so-called new mathematics. A well-known mathematics educator feels that introducing formal rules of symbolic logic below the college level is against the weight of mathematic judgment about how clarity of thought is developed.[2] On the other hand, changes in the teaching of elementary school mathematics may be viewed as a result of the maturing of mathematics itself and its more intimate relation to man's everyday activities.[3]

PROPOSALS FOR CHANGE

A number of individuals and groups have conducted long-term experiments and have written sample textbooks and monographs on elementary mathematics. Under consideration are programs which push some topics down one or more grades, change the way topics are presented, and introduce new material in the elementary grades.

The School Mathematics Study Group has done considerable work

[1] Herbert F. Spitzer, Statement in section "Teaching Modern Mathematics," *NEA Journal*, 51 (November, 1962), 45.

[2] Saunders MacLane, "The Reform Has Been Oversold," *Ibid.*, 51 (November 1962), 45–46.

[3] Irving Adler, "The Changes Taking Place in Mathematics," *The Leadership Role of State Supervisors of Mathematics* (Washington, D.C.: U.S. Department of Health, Education, and Welfare, 1962), 11–27.

on new mathematical topics for the junior high grades and the elementary grades. The SMSG had a director and an advisory committee composed of mathematicians, high school mathematics teachers, educational consultants, and others. Projects included the development of experimental units for use at the elementary and junior high levels and the writing of sample textbooks and monographs on various mathematical topics. Many schools are using these materials to supplement regular textbooks and programs. Some are using SMSG publications as the textbook in grades seven and eight.

The University of Illinois Arithmetic Project was a five-year experiment in arithmetic for grades one through six. The purpose of the project was developing better content for elementary mathematics and improved ways of presenting it to children. Introduction of more sophisticated mathematical concepts at a younger age seems to be indicated by the results. Real curriculum changes may be a result of this project.

The University of Maryland Mathematics Project has been concerned with grades seven and eight. Its study has been guided by an advisory committee representing the fields of mathematics, science, engineering, and education at the university, as well as the U.S. Office of Education, Maryland State Department of Education, and adjacent public school systems. The staff of the project and teachers of experimental classes near the university have developed experimental materials for seventh and eighth-grade arithmetic. These materials have been used in a large number of experimental classes. Results are available to persons and groups interested in experimenting with mathematics programs at these grade levels.

Geometry for the primary grades has been a subject for experimentation at Stanford by Hawley and Suppes. A text containing simple materials on lines, circles, and constructions was developed and taught to several thousand primary-grade children. Children learned quickly. Work is continuing in this area. Suppes is also experimenting with materials on elementary set theory. First-grade children are learning about operations with sets along with ideas on numbers. Concrete work with sets may help to make number ideas more meaningful.

The Madison Project, operating in New York and Connecticut, has been concerned with improving mathematics programs from grade four through the junior high grades. A two-year algebra program, suitable for children down to grade three, has been developed along with workbook materials. The program has met with success in a

from a few dollars to thousands of dollars. It has developed, however, that the heart of the process is not the machine but the program, or text material, used with it. Consequently, a steadily increasing amount of attention has been given to the development and use of good programs.

One type of program, the so-called linear variety, presents a concept by breaking it down into a large number of very small steps. As a student advances through the program, the sequence is essentially this: (1) a small bit of information is presented; (2) he is confronted with a question of a type that, if he has mastered the material in earlier frames, he should be able to answer correctly; (3) having arrived at his answer or response, he turns a page or activates a machine so that he now sees a confirmation frame. Because of the minute steps with which he moves forward, the student usually is able to answer correctly. This means that he encounters success, or is rewarded, frequently as he advances through the program.

Another type of programing uses the "scrambled text." In this program, the student follows the first two steps as just listed, but the confirmation process is different. Let us say that, after a new bit of information is presented, the student is confronted with a four-option multiple-choice question based upon it. Each answer carries a reference to another page of the text. If the student picks the correct response, the page to which he is referred tells him his answer is correct, explains why it is correct, then presents the next step in the mastery process. If, however, he picks an incorrect response, the page to which he is referred tells him his answer is wrong, explains the misconceptions that were probably involved, and refers him back to his original frame to try again.

SOME USES

You probably have concluded by now that a program is not designed to replace a teacher but that it does have merit as an aid in teaching. In addition to the psychological principle of reinforcement, this method involves certain other features that are potentially helpful; thus (1) each individual can progress at his own rate; (2) each is continually informed as to his progress; (3) it is a good way for a student to self-administer drill.

When we view the exceptional learner — either the high-ability or low-ability student — as one whose learning pattern is a departure from

the norm, we may see an opportunity to make effective use of an individualized approach. Programed arithmetic materials may turn out to be very helpful in individualizing our teaching for the exceptional learner. There may, however, be some danger of doing just the opposite. These materials may be misused in such a way as to give arithmetic instruction a routine sameness for all children.

Publicity surrounding the introduction of teaching machines and programed materials makes it necessary to proceed carefully in their use. So important is this matter that a joint committee of the American Educational Research Association, the American Psychological Association, and the Department of Audio-Visual Instruction, NEA, has issued a statement on the use of self-instructional materials and devices. The committee noted that these materials and devices represent a potential contribution of great importance to American education. The committee felt, however, that information to evaluate these materials was necessary and suggested some tentative guidelines. The following have been adapted from the statement of the committee:

1. Teaching is done by a program of instructional materials and not by a teaching machine. Any evaluation of a teaching machine requires an assessment of the availability and quality of programs for each type of machine.
2. Not all programs fit all machines. Only programs which fit a particular machine can be considered available for use with it.
3. Programs labeled with the name of a particular subject (arithmetic, for example) vary widely in content and instructional objectives. They should be examined to determine what the student is required to do and whether student responses reflect the competences which the educator wishes to achieve.
4. Just any set of question-and-answer material does not make a self-instructional program. One type of program proceeds by small steps requiring frequent student responses. Items in such a program must be carefully designed so that students perform the operations each item was meant to teach.
5. Self-instructional materials are designed to adapt to individual differences by allowing each student to proceed at his own rate. Some types of such materials further adapt by providing "branching" to alternate materials based on questions which diagnose the student's needs.

6. Most self-instructional materials provide a record of the student's responses. A purchaser should inquire about the extent of revision of materials based on student response.
7. The effectiveness of a self-instructional program should be based on what students learn and remember from the program. A purchaser should find out what data are available and the conditions under which the data were obtained.
8. Experimentation with self-instructional materials and devices should be encouraged before any large-scale adoption.

ARITHMETIC PROGRAMS

A number of companies are now publishing arithmetic programs for use in teaching machines. Some companies have available the so-called scrambled textbooks on various topics in arithmetic. The following list (continued on page 398) gives some of these companies and their materials.

Company	*Materials*
Astra Corporation 31 Church Street New London, Conn.	Machine Programs: 390 Arithmetic Facts Introduction to and Functions of Fractions Introduction to and Functions of Decimals Percentages and Denominate Numbers
TMI-Grolier 525 Lexington Avenue New York 22, N.Y.	Machine Programs: Addition and Subtraction Multiplication and Division Decimal Numbers Fractions
Devereaux Teaching Aids Box 717 Devon, Pa.	Programed Workbooks: Lower Primary Arithmetic Upper Primary Arithmetic Addition and Subtraction Multiplication and Division Fractions (all for use in special education classes)
General Programmed Teaching Corporation 1719 Girard NE Albuquerque, New Mexico	Machine Programs: Fractions I Fractions II Ratios, Proportions, and Percentages

Company	*Materials*
Field Enterprises Educational Corporation Merchandise Plaza Chicago 54, Ill.	Machine Programs: Reviewing Addition Facts; Reviewing Subtraction Facts; Reviewing Multiplication Facts; Reviewing Division Facts; Reviewing Fractions; Reviewing Decimals

The foregoing is only a partial listing of companies publishing programs in arithmetic. The authors have made no evaluation of these programs and do not intend this list to be a recommendation of these programs. More complete lists of programs and publishers are available.[4]

C. Issues in Teaching Elementary School Mathematics

Despite the tremendous amount of research that has been and is being done in the teaching of elementary mathematics, many important issues are still unsettled. Indeed, one cannot imagine a time when all such questions will have been answered, since new ones are constantly arising.

Many of the major issues in elementary mathematics can be grouped under three general questions: (1) What shall be taught and when? (2) How shall it be taught? (3) By whom shall it be taught?

WHAT SHALL BE TAUGHT AND WHEN?

As has been mentioned, the coming of the graded elementary school required the assignment of mathematics topics to the various grade levels. This was done on a somewhat arbitrary basis. After a considerable amount of study, testing, and experience, however, the grade placement of topics achieved some degree of stability. This condition prevailed, over the objections of certain critics, until shortly after 1950, when more and more questions began to arise regarding content generally, as well as grade placement of content.

Content. Drastic departures from traditional arithmetic content have been sponsored by a variety of groups, most of which were mentioned earlier. As is true of most innovations, the advocates saw the new content as being patently superior to the traditional type, whereas critics saw little good in the new material. Evaluation of the new topics will actually require a considerable period of time and a great deal of research. Hence, the study of content continues to be an important area of research.

Grade placement. Although it is assumed that many teachers will vary content according to the ability of their students, there is no escaping the fact that arithmetic content in a particular grade is greatly influenced by the textbook or texts and courses of study. Certain topics have been up and down the grade scale for many years and still have not stabilized. Indeed, the placement of all topics is a matter of constant study. For example, some programs complete the presentation of multiplication facts in fourth grade. But couldn't this be done in third grade? Would learning be more permanent if this phase were completed in fifth grade?

Some attention has been given to the use of geometric tools in the lower elementary grades. The proponents argue that small children can use such instruments successfully. Skeptics agree but wonder whether it is worth the time and effort to teach these topics to young children. Indeed, this illustrates the pattern that holds for many areas of study: some say it can and should be done; others wonder whether it is that which is most needed at the level in question.

HOW SHALL IT BE TAUGHT?

The whole area of method in elementary mathematics is replete with unanswered questions. A few of these are:

1. What is the role of drill? Psychological studies have indicated that prolonged periods of drill are of doubtful value. On the other hand, many elementary teachers insist that there is no substitute for well-directed drill in certain phases of their work. The role of self-administered drill through the use of programed materials is and will probably continue to be a major issue in elementary mathematics teaching.
2. What is the best use of concrete materials in teaching elementary mathematics? The widespread usage of counters and related

objects, particularly in introducing certain topics in arithmetic, is evidence of teacher attitude. On the other hand, one might well ask why we should make such materials available to children but later deny them the use of these same materials.

3. How much emphasis should be given to teaching for understanding? Were earlier methods, based largely upon memorizing and applying rules, entirely unsuccessful? What shall we do with those students who have difficulty in understanding a process but are quite successful in applications, and vice versa? Related to this is the never-ending difficulty of trying to teach problem solving. Few areas in the teaching of elementary mathematics need systematic study more than does this one.

The list of issues dealing with "How Shall It Be Taught" could be continued almost indefinitely. The ones just mentioned, however, illustrate the type of questions that await answers.

BY WHOM SHOULD IT BE TAUGHT?

The difficulties encountered in trying to prepare an elementary teacher to function in all subject areas are well known. One point of view might be that we should encourage some degree of subject area specialization among elementary faculties. The proponents of this point of view cite the fact that this method is almost universal in secondary teaching. Others, however, question the ability of small children to adapt to a departmentalized situation. Intermediate approaches, such as team teaching, are being explored. There does not appear to be any major trend away from the traditional classroom teaching situation in the elementary school.

A related issue is: What type of mathematics training should be given the prospective elementary teacher? There are many points of view. The elementary curriculum in the typical teacher-training program is very crowded. The addition of more college mathematics courses would necessitate the removal of courses in other subjects which, in turn, would probably create problems elsewhere. So, in all likelihood, the long-standing debate as to the best pattern of training for elementary teachers will continue.

Something to Think About

1. On the basis of articles in professional journals, describe the usage of programed materials in teaching elementary school mathematics.

2. Compare a new edition of an elementary school mathematics textbook with the edition that preceded it. How has the so-called modern movement influenced the content of the new edition?

3. What seem to be the current beliefs about giving prospective elementary school teachers the necessary mathematics background during their college work?

4. How would you explain the teaching of set theory in the elementary grades to a skeptical parent?

5. Using *The Arithmetic Teacher* and other sources, prepare a report on one of the various experimental projects that has affected arithmetic content and grade placement of topics.

6. Select a problem or issue confronting elementary mathematics teachers. Prepare a report outlining several views on the problem or issue.

Selected References

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Each issue contains reports on experimental projects and research affecting content, materials, and methods in arithmetic.

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